QUASIDIAGONALITY OF
NUCLEAR C*-ALGEBRAS

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ABSTRACT. We prove that faithful traces on separable and nuclear C*-algebras in the UCT class are quasidiagonal. This has a number of consequences. Firstly, by results of many hands, the classification of unital, separable, simple and nuclear C*-algebras of finite nuclear dimension which satisfy the UCT is now complete. Secondly, our result links the finite to the general version of the Toms–Winter conjecture in the expected way and hence clarifies the relation between decomposition rank and nuclear dimension. Finally, we confirm the Rosenberg conjecture: discrete, amenable groups have quasidiagonal C*-algebras.

INTRODUCTION

Quasidiagonality was first introduced by Halmos for sets of operators on Hilbert space; see [35, Section 4]. An abstract C*-algebra is called quasidiagonal if it has a faithful representation that is quasidiagonal, i.e., for which there is an approximately central net of finite rank projections converging strongly to the unit (see [65, 72], or [9], or [11, Chapter 7]). In [71], Voiculescu characterised quasidiagonality in terms of the existence of almost isometric and almost multiplicative completely positive contractive (c.p.c.) maps into finite dimensional C*-algebras. It was observed in [72, 2.4] that unital quasidiagonal C*-algebras always have traces (see also [11, Proposition 7.1.6]). Traces which can be witnessed by quasidiagonal approximations are called quasidiagonal (see Definition 1.2 below); these were introduced and systematically investigated in [10].

In Hadwin's paper [34] quasidiagonality was linked to nuclearity of C*-algebras. The latter can be expressed in terms of c.p.c. approximations through finite dimensional C*-algebras; it is in many respects analogous to amenability for discrete groups. This connection was further exploited in [5]; it also features prominently in Elliott's programme to classify simple nuclear C*-algebras by K-theoretic data.

In [60], Rosenberg and Schochet established what they called the universal coefficient theorem (UCT), relating Kasparov's bivariant KK-theory to homomorphisms between K-groups. They showed that the UCT holds for all separable nuclear C*-algebras which are KK-equivalent to abelian ones. It remains an important open problem dating back to [60] whether this class in fact contains all separable nuclear C*-algebras. At first sight

the UCT does not seem related to quasidiagonality, but one link was noted by L. Brown in his work [8] on universal coefficient theorems for Ext; see [72, 2.3]. Our main result provides a new connection.

Theorem A. *Let A be a separable, nuclear C^*-algebra which satisfies the UCT. Then every faithful trace on A is quasidiagonal.*

Before outlining the proof at the end of this introduction, let us describe some consequences. First, as the existence of a faithful quasidiagonal trace entails quasidiagonality, our result shows that many nuclear C^*-algebras are quasidiagonal. Blackadar and Kirchberg in [5] analysed the relation between quasidiagonality and other approximation properties, introducing the notions of generalised inductive limits and of MF and NF algebras. They asked whether every stably finite nuclear C^*-algebra is quasidiagonal (now known as the Blackadar–Kirchberg problem). Theorem A gives a positive solution under the additional hypotheses that there is a faithful trace and the UCT holds. We obtain:

Corollary B. *Every separable, nuclear C^*-algebra in the UCT class with a faithful trace is quasidiagonal. In particular, the Blackadar–Kirchberg problem has an affirmative answer for simple C^*-algebras satisfying the UCT.*

In [34], Hadwin asked which locally compact groups have (strongly) quasidiagonal C^*-algebras. In the appendix to [34], Rosenberg showed that any countable discrete group G with quasidiagonal left regular representation is amenable. As pointed out in [11, Corollary 7.1.17], the argument does not really depend on the specific representation, and G is amenable provided $C_r^*(G)$ is quasidiagonal. The converse to this statement – is $C_r^*(G)$ quasidiagonal if G is amenable – has received substantial attention over the decades and is now commonly referred to as the Rosenberg conjecture; cf. [72, Section 3] and [11, Section 7.1]. Recently, Carrion, Dadarlat, and Eckhardt gave a positive answer for groups which are locally embeddable into finite groups (LEF groups) in [13]. Even more recently, Ozawa, Rørdam, and Sato handled the case of elementary amenable groups and extensions of LEF groups by elementary amenable groups in [53]. In this paper we settle the matter, as a consequence of Theorem A:

Corollary C. *Let G be a discrete, amenable group. Then $C_r^*(G)$ is quasidiagonal.*

Using results from classification, we actually get the stronger statement that $C_r^*(G)$ is AF-embeddable if G is countable, discrete, and amenable; see Corollary 6.6 (see also [72, Section 4] and [11, Section 8] for a discussion of AF-embeddability and relations to quasidiagonality).

Note that the approach of [53] uses the full strength of Elliott's programme in the simple and monotracial case (which is remarkable, since group C^*-algebras themselves are not expected to be directly accessible to classification). Our argument also has a classification component (albeit of a more basic sort), namely through the stable uniqueness technology introduced by Lin in [43, 45], (we use the version of Dadarlat and Eilers in [21]), of which more is said below.

The first prominent entry of quasidiagonality into Elliott's classification programme was in the purely infinite case, in the proof of Kirchberg's $\mathcal{O}_2$-embeddability theorem [38], via Voiculescu's remarkable result on homotopy invariance of quasidiagonality [71].

The relevance of quasidiagonality in the stably finite case goes back to Popa's article [54], which uses local quantisation to establish an internal approximation property for simple quasidiagonal C*-algebras with sufficiently many projections. This idea triggered Lin's definition of tracially approximately finite dimensional (TAF) algebras, which then made it possible to come up with the first stably finite analogue of Kirchberg–Phillips classification; see [44].

The potential relevance of quasidiagonality of traces to Elliott's programme was highlighted well ahead of its time by N. Brown in [10, Section 6.1]. This foresight has come to fruition very recently through the entry of quasidiagonal traces in the noncommutative covering dimension calculations of [6]. The main result of [29] (which in turn relies on [30, 55, 73] to reduce to the classification theorem of [32], a result which builds on a large body of work over decades) classifies separable, unital, simple, nuclear C*-algebras with finite nuclear dimension (the noncommutative notion of covering dimension from [79]) in the UCT class under the additional assumption that all traces are quasidiagonal. Theorem A removes this assumption, and so completes the classification of these algebras. Since finite nuclear dimension is precisely the "finitely coloured" or higher dimensional version of being approximately finite dimensional (AF), the following corollary can be viewed as the C*-analogue of the classification of hyperfinite von Neumann factors.

Corollary D. *Let A and B be separable, unital, simple and infinite dimensional C*-algebras with finite nuclear dimension which satisfy the UCT. Then A is isomorphic to B if and only if A and B have isomorphic Elliott invariants.*

We wish to point out that our contribution is just the final piece in a long-time endeavour. Also, it only affects the stably finite case of Elliott's programme – the purely infinite case was dealt with by Kirchberg and Phillips in the 90's. We deliberately state Corollary D without making this distinction to emphasise that the two strands now run completely parallel.

In the situation of the corollary, one can extract much more refined information. For example, the C*-algebras are inductive limits of nice building block algebras with low topological dimension (subhomogeneous algebras in the stably finite case and Cuntz algebras over circles in the purely infinite situation; see [58]). Moreover, they have nuclear dimension 1 and decomposition rank ∞ (see [42]) when they are purely infinite (under the hypotheses of the corollary, this is equivalent to the absence of traces); in the stably finite case (i.e., in the presence of traces) their decomposition rank is at most 2 (in many cases, and probably always, it is at most 1). The nuclear dimension hypothesis in Corollary D is known to hold for $\mathcal{Z}$-stable C*-algebras whose trace spaces are empty or have compact extreme boundaries; see [6]. (Here, $\mathcal{Z}$-stability is tensorial absorption of the Jiang–Su algebra $\mathcal{Z}$ – see [37] – which is a C*-algebraic analogue of a von Neumann algebra being McDuff.) In the case of at most one trace we obtain a particularly clean statement,

which again encapsulates both the finite and the infinite case (and which unlike Corollary D does not rely on the as yet unpublished results of [6], [29] and [32]):

Corollary E. *Separable, simple, unital, nuclear* C*-*algebras in the UCT class with at most one trace are classified up to $\mathcal{Z}$-stability by their ordered K-theory.*

By classification up to $\mathcal{Z}$-stability, we mean that any isomorphism between the Elliott invariants (which reduce to ordered K-theory in the monotracial case and to just K-theory in the purely infinite situation) of $A \otimes \mathcal{Z}$ and $B \otimes \mathcal{Z}$ lifts to an isomorphism between $A \otimes \mathcal{Z}$ and $B \otimes \mathcal{Z}$. There are many important classes of C*-algebras that are automatically $\mathcal{Z}$-stable, and there are highly useful criteria to check this property. Even without $\mathcal{Z}$-stability we obtain a nontrivial statement, since one can simply tensor by $\mathcal{Z}$ and determine the isomorphism class of the resulting C*-algebra in terms of the invariant.

We will discuss these and other consequences of Theorem A – including its relation to the Toms–Winter conjecture – in greater detail in Section 6.

The rest of this introduction is devoted to an outline of the proof of Theorem A, so fix a separable, unital and nuclear C*-algebra A in the UCT class, and a faithful trace τ_A on A. We seek an embedding Ψ of A into the ultrapower, $\mathcal{Q}_\omega$, of the universal UHF algebra inducing τ_A (i.e. $\tau_A = \tau_{\mathcal{Q}_\omega} \circ \Psi$).

Our starting point is the order zero tracial quasidiagonality result from [62, Section 3], a consequence of the uniqueness of the injective II_1 factor. From this one obtains a kind of "2-coloured quasidiagonality of τ_A", which we express in terms of homomorphisms from cones: there exist two *-homomorphisms[1] $\grave{\Phi} : C_0([0,1), A) \to \mathcal{Q}_\omega$ and $\acute{\Phi} : C_0((0,1], A) \to \mathcal{Q}_\omega$, which both induce the trace $f \mapsto \int_0^1 \tau_A(f(t))dt$ and whose scalar parts $C_0([0,1), \mathbb{C}1_A)$ and $C_0((0,1], \mathbb{C}1_A)$ are the restrictions of a common unital *-homomorphism $\Theta : C([0,1]) \to \mathcal{Q}_\omega$. The details of how to obtain these cone *-homomorphisms from [62] are the subject of Section 2.

To construct the desired *-homomorphism Ψ from $\grave{\Phi}$ and $\acute{\Phi}$ we use a "stable uniqueness across the interval argument" (see [30] for an argument of a similar flavour). Imagine that we can find some (presumably large) $n \in \mathbb{N}$ such that for any open interval $J \subseteq (0,1)$,

$$\grave{\Phi}|_{C_0(J,A)}^{\oplus n} \text{ and } \grave{\Phi}|_{C_0(J,A)}^{\oplus(n-1)} \oplus \acute{\Phi}|_{C_0(J,A)}^{\oplus 1} \text{ are unitarily equivalent; and}$$

$$(0.1) \quad \acute{\Phi}|_{C_0(J,A)}^{\oplus n} \text{ and } \acute{\Phi}|_{C_0(J,A)}^{\oplus(n-1)} \oplus \grave{\Phi}|_{C_0(J,A)}^{\oplus 1} \text{ are unitarily equivalent,}$$

as maps into $\mathcal{Q}_\omega \otimes M_n$. Write $N := 2n$ and divide $[0,1]$ into N overlapping subintervals $I_1, \ldots, I_N$ as painted in Figure 1.[2]

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[1]The notation $\grave{\Phi}$ and $\acute{\Phi}$ is designed to indicate the orientation of the cones $C_0([0,1))$ and $C_0((0,1])$ respectively through the appearance of canonical positive contractions generating these cones.

[2]In the actual proof, we define I_1 and I_N to be open, and introduce additional intervals I_0, I_{N+1} and corresponding maps.
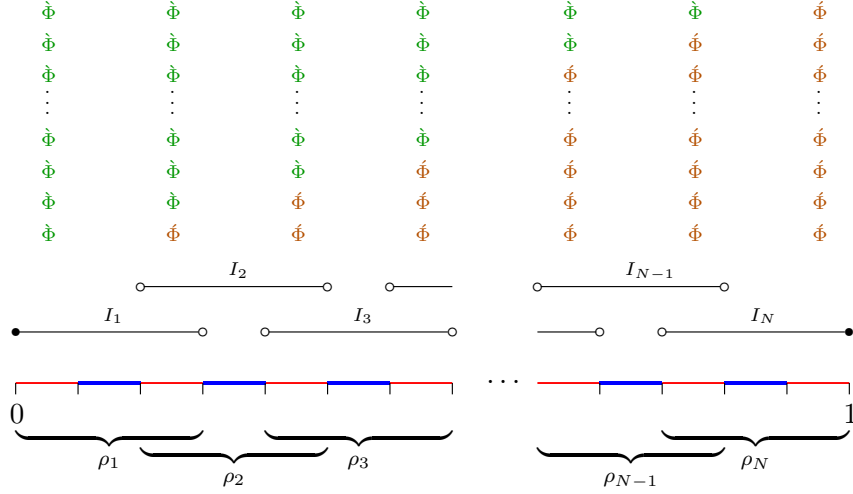
FIGURE 1. Stable uniqueness across the interval

In this way, on each closed blue middle third of I_i

$$(0.2) \qquad \grave{\Phi}|_{...}^{\oplus(N-i+1)} \oplus \acute{\Phi}|_{...}^{\oplus(i-1)} \text{ and } \grave{\Phi}|_{...}^{\oplus(N-i)} \oplus \acute{\Phi}|_{...}^{\oplus i}$$

are unitarily equivalent as maps into $\mathcal{Q}_\omega \otimes M_N$ (using either the first or second line of (0.1), depending on whether $i \leq n$ or not). From this unitary equivalence, we can "patch" across each of the intervals I_i, to produce *-homomorphisms ρ_i defined on $C_0(I_i, A)$ which are compatible with Θ and agree with the two maps in (0.2) on the left hand and right red thirds of I_i. This is achieved using the additional space given by a 2-fold matrix amplification, so that each ρ_i maps $C_0(I_i, A)$ into $\mathcal{Q}_\omega \otimes M_N \otimes M_2$ and restricts to

$$(0.3) \qquad \begin{pmatrix} \grave{\Phi}|_{...}^{\oplus(N-i+1)} \oplus \acute{\Phi}|_{...}^{\oplus(i-1)} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \grave{\Phi}|_{...}^{\oplus(N-i)} \oplus \acute{\Phi}|_{...}^{\oplus i} & 0 \\ 0 & 0 \end{pmatrix}$$

on the left and right hand thirds of I_i respectively. Additionally, the construction is such that ρ_i induces the tracial functional $h \mapsto \frac{1}{2} \int_{I_i} \tau_A(h(t)) \, dt$; the factor of $\frac{1}{2}$ arising from the two-fold amplification. In particular ρ_i and ρ_{i+1} agree on $I_i \cap I_{i+1}$. Then, taking a partition of unity $(f_i)_i$ for $C([0,1])$ subordinate to $(I_i)_i$, $\Psi(a) := \sum_{i=1}^{N} \rho_i(f_i \otimes a)$ defines a *-homomorphism into $\mathcal{Q}_\omega \otimes M_N \otimes M_2$ inducing $\frac{1}{2}\tau_A$. This would suffice to establish quasidiagonality of τ_A.

Now we turn to whether such an n satisfying (0.1) can be found. The short answer is that it cannot, but working throughout with the greater flexibility of approximate unitary equivalence on finite sets up to specified tolerances, a small modification of a "stable uniqueness theorem" of Dadarlat and Eilers from [21] – which was motivated by, and both reproves and generalises that of Lin [43] (see also [45]) – gives the required n.

Roughly speaking, stable uniqueness theorems ensure that nuclear *-homomorphisms $\phi, \psi : C \to D$ which agree at the level of KK become approximately unitary equivalent after taking a direct sum with an amplification of a suitable map $\iota : C \to D$. Somewhat more precisely, given a finite subset $\mathcal{G}$ of C and tolerance $\delta > 0$, there exists some $n \in \mathbb{N}$ such that $\phi \oplus \iota^{\oplus n}$

is approximately unitarily equivalent to $\psi \oplus \iota^{\oplus n}$ on $\mathcal{G}$ up to δ (see [21, Theorem 4.5] and Theorem 3.1 below for the precise statement). As explained in [21], for many applications, including ours, it is of particular importance to know when n can be chosen to depend only on $\mathcal{G}$ and δ and not on the particular maps ϕ, ψ and ι. Using the UCT, Dadarlat and Eilers achieved such a result, which has subsequently seen wide application, by means of a sequence of counterexamples argument for simple and unital A (see [21, Theorem 4.12]). The UCT enters crucially to control the KK-classes of the sequence of counterexamples.

In our case the domain C is nonsimple, so we give a slight generalisation of [21, Theorem 4.12] to allow for this, using "controlled fullness" in place of simplicity. In nice cases (including the situation of Theorem A), this leads to a stable uniqueness theorem where n depends on $\mathcal{G}$ and δ and the tracial data of ι, but not on further properties of ι, nor on ϕ and ψ. We set out this generalisation in Section 3. A similar sort of result can be found in [45].

Thus, we fix a canonical domain algebra $C \cong C_0((\frac{1}{3}, \frac{2}{3}), A)^\sim$ which, after an appropriate rescaling, will stand in for $C_0(J, A)^\sim$ (J being a small interval), and obtain an approximate version of (0.1) for $\mathcal{G}$ and δ (which in turn come from an approximate version of the patching lemma). However, as $|J|$ gets smaller, the estimates on the controlled fullness of the resulting maps

$$(0.4) \qquad C \xrightarrow{\ \cong\ } C_0(J, A)^\sim \xrightarrow{\acute{\Phi}|_{\widetilde{C_0(J,A)}}, \ \grave{\Phi}|_{\widetilde{C_0(J,A)}}} \mathcal{Q}_\omega$$

will get worse. This is rectified using projections $(q_J)_J$ in $\mathcal{Q}_\omega$ (constructed in Lemma 1.6) such that the maps in (0.4) take values in the corner $q_J \mathcal{Q}_\omega q_J$. With respect to the unique normalised trace on this corner, the tracial behaviour of these maps becomes independent of J so that our controlled stable uniqueness theorem applies (agreement of the maps in (0.4) on total K-theory is essentially automatic, as they are constructed from contractible maps). Finally, by exploiting the inherent orthogonality of ρ_i and ρ_j whenever $I_i \cap I_j = \varnothing$, we ensure that the final approximate multiplicativity estimates for Ψ do not depend on n.

Full details of the stable uniqueness across the interval argument are given in Section 5 with the approximate patching result in Section 4.

1. Quasidiagonality, $\mathcal{Q}_\omega$, and the UCT

In this section we record a number of preliminary facts regarding quasidiagonality, quasidiagonal traces, the ultrapower of the universal UHF algebra and the universal coefficient theorem.

Let A be a C*-algebra. Denote the positive cone of A by A_+, and the set of positive contractions by A_+^1. For $a, b \in A$ and $\varepsilon > 0$, we write $a \approx_\varepsilon b$ to mean $\|a - b\| \leq \varepsilon$ (and we use the respective notation for real or complex numbers). For a unitary $u \in A$, we let $\mathrm{Ad}(u)$ denote the automorphism of A defined by $\mathrm{Ad}(u)(a) := uau^*$, $a \in A$.

A *tracial functional* is a positive functional $\tau : A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$ for every $a, b \in A$ and for us a *trace* is a tracial functional that is normalised, i.e., a state. We write $T(A)$ for the collection of all traces on A. The C*-algebra of $k \times k$ complex matrices is denoted M_k; we write τ_{M_k} for its unique trace, and Tr_k for the canonical tracial functional on M_k satisfying $\mathrm{Tr}_k(1_k) = k$.

We now recall the central notions of this paper: quasidiagonality and quasidiagonal traces. (Rather than the original definition of quasidiagonality in terms of a quasidiagonal faithful representation, we use as our definition a formulation shown to be equivalent by Voiculescu in [71, Theorem 1].) We make these definitions for unital C*-algebras, the primary context of the paper; see Remark 1.5 for the non-unital case.

Definition 1.1. A unital C*-algebra A is *quasidiagonal* if, for every finite subset $\mathcal{F}_A$ of A and $\varepsilon > 0$, there exist a matrix algebra M_k and a unital completely positive (u.c.p.) map $\psi : A \to M_k$ such that

$$(1.1) \qquad \psi(ab) \approx_\varepsilon \psi(a)\psi(b), \quad a, b \in \mathcal{F}_A, \quad \text{and}$$

$$(1.2) \qquad \|\psi(a)\| \approx_\varepsilon \|a\|, \quad a \in \mathcal{F}_A.$$

Definition 1.2 ([10, Definition 3.3.1]). Let A be a unital C*-algebra. A trace $\tau_A \in T(A)$ is *quasidiagonal* if for every finite set $\mathcal{F}_A$ of A and $\varepsilon > 0$, there exists a matrix algebra M_k and a u.c.p. map $\psi : A \to M_k$ such that

$$(1.3) \qquad \psi(ab) \approx_\varepsilon \psi(a)\psi(b), \quad a, b \in \mathcal{F}_A, \quad \text{and}$$

$$(1.4) \qquad \tau_{M_k}(a) \approx_\varepsilon \tau_A(a), \quad a \in \mathcal{F}_A.$$

Note that quasidiagonality of a trace τ_A does not mean the same thing as quasidiagonality (as a set of operators, in Halmos' sense) of $\pi_{\tau_A}(A)$, where π_{τ_A} is the GNS representation corresponding to τ_A.

We now turn to a standard characterisation of quasidiagonality in terms of embeddings. Write $\mathcal{Q}$ for the universal UHF algebra. Fix, throughout the paper, a free ultrafilter ω on $\mathbb{N}$, and let $\mathcal{Q}_\omega$ be the ultrapower of $\mathcal{Q}$, defined by

$$(1.5) \qquad \mathcal{Q}_\omega := \ell^\infty(\mathcal{Q})/\{(x_n) \in \ell^\infty(\mathcal{Q}) \mid \lim_{n \to \omega} \|x_n\| = 0\}.$$

When there is no prospect of confusion, we often work with a representative of an element x in $\mathcal{Q}_\omega$, namely a lift of x to some element in $\ell^\infty(\mathcal{Q})$ as in (1.6) and Lemma 1.6 below. In particular, recall that all projections can be represented by sequences of projections, and unitaries by sequences of unitaries.

The unique trace $\tau_\mathcal{Q}$ on $\mathcal{Q}$ induces a trace $\tau_{\mathcal{Q}_\omega}$ on $\mathcal{Q}_\omega$ by

$$(1.6) \qquad \tau_{\mathcal{Q}_\omega}(x) := \lim_\omega \tau_\mathcal{Q}(x_n), \quad x \in \mathcal{Q}_\omega \text{ is represented by } (x_n) \in \ell^\infty(\mathcal{Q}).$$

By [50, Lemma 4.7] (see also [52, Theorem 8]), $\tau_{\mathcal{Q}_\omega}$ is the unique trace on $\mathcal{Q}_\omega$. The following two observations are consequences of the standard fact that for any nonzero projection $q \in \mathcal{Q}$, $q\mathcal{Q}q \cong \mathcal{Q}$.

Proposition 1.3. *(i) For any nonzero projection $q \in \mathcal{Q}_\omega$, $q\mathcal{Q}_\omega q \cong \mathcal{Q}_\omega$.*

(ii) For any projection $q \in \mathcal{Q}_\omega$ with $\tau_{\mathcal{Q}_\omega}(q) > 0$, there exist $k \in \mathbb{N}$ and a (not necessarily unital) embedding $\theta : \mathcal{Q}_\omega \to q\mathcal{Q}_\omega q \otimes M_k$ such that $\theta(qxq) = qxq \otimes e_{11}$ for $x \in \mathcal{Q}_\omega$.

Proof. (i) Given a nonzero projection $q \in \mathcal{Q}_\omega$, we can find a representative (q_n) of q such that each q_n is a nonzero projection in $\mathcal{Q}$. For each n, fix an isomorphism $\theta_n : \mathcal{Q} \to q_n\mathcal{Q}q_n \subset \mathcal{Q}$. Then the sequence (θ_n) induces an injective *-homomorphism $\theta : \mathcal{Q}_\omega \to \mathcal{Q}_\omega$ with image $q\mathcal{Q}_\omega q$.

(ii) Given such a q, fix $k \in \mathbb{N}$ with $\frac{1}{k} < \tau_{\mathcal{Q}_\omega}(q)$, and fix a representative sequence (q_n) of q by projections in $\mathcal{Q}$ with $\tau_{\mathcal{Q}}(q_n) > \frac{1}{k}$ for all n. Since $k\tau_{\mathcal{Q}}(q_n) > 1$, elementary properties of $\mathcal{Q}$ provide embeddings $\theta_n : \mathcal{Q} \to q_n\mathcal{Q}q_n \otimes M_k$ with $\theta_n(q_nxq_n) = q_nxq_n \otimes e_{11}$ for $x \in \mathcal{Q}$. The embedding $\theta : \mathcal{Q}_\omega \to q\mathcal{Q}_\omega q \otimes M_k$ induced by the sequence (θ_n) then has the specified property. $\qquad\qquad\square$

We now record characterisations of quasidiagonality and quasidiagonal traces in terms of maps into $\mathcal{Q}_\omega$. The condition in (iic) below is exactly the form we will use to prove Theorem A.

Proposition 1.4. *Let A be a separable, unital and nuclear C*-algebra.*

(i) Then A is quasidiagonal if and only if there exists a unital embedding $A \hookrightarrow \mathcal{Q}_\omega$.

(ii) For a trace $\tau_A \in T(A)$, the following statements are equivalent:
(a) τ_A is quasidiagonal;
*(b) there exists a unital *-homomorphism $\theta : A \to \mathcal{Q}_\omega$ such that $\tau_{\mathcal{Q}_\omega} \circ \theta = \tau_A$;*
(c) there exists $\gamma \in (0,1]$ such that for every finite subset $\mathcal{F}_A \subset A$ and $\varepsilon > 0$ there is a completely positive map $\phi : A \to \mathcal{Q}_\omega$ such that

$$(1.7) \qquad\qquad \phi(ab) \approx_\varepsilon \phi(a)\phi(b), \quad a, b \in \mathcal{F}_A, \ \text{and}$$

$$(1.8) \qquad\qquad \tau_{\mathcal{Q}_\omega} \circ \phi(a) = \gamma\tau_A(a), \quad a \in \mathcal{F}_A.$$

In particular if A has a faithful quasidiagonal trace, then A is quasidiagonal.[3]

Proof. (i) is well-known; we recall the argument as it will be reused in (ii). By the Choi–Effros lifting theorem [14, Theorem 3.10], a unital embedding of A into $\mathcal{Q}_\omega$ can be lifted to a sequence of c.p.c. maps into $\mathcal{Q}$, which must be approximately multiplicative and approximately isometric (along ω). Following these maps by expectations onto sufficiently large unital matrix subalgebras of $\mathcal{Q}$ gives the maps required by Definition 1.1. Conversely, by separability of A, Definition 1.1 provides a sequence $\theta_n : A \to M_{k_n} \subset \mathcal{Q}$ of approximately multiplicative and approximately isometric u.c.p. maps. The induced map is a unital embedding $A \hookrightarrow \mathcal{Q}_\omega$.

The equivalence of (iia) and (iib) proceeds in exactly the same fashion as the proof of (i). It is clear that (iib) implies (iic) (with $\gamma = 1$). Conversely, if

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[3]Using a variant of this proposition which involves maps into $\mathcal{Q}_\omega$ with c.p. liftings to $\ell^\infty(\mathcal{Q})$, this last statement holds for not-necessarily nuclear C*-algebras.

condition (iic) holds, then a standard reindexing argument (see for example [6, Section 1.3]) gives a *-homomorphism $\theta : A \to \mathcal{Q}_\omega$ such that $\tau_{\mathcal{Q}_\omega}(\theta(a)) = \gamma \tau_A(a)$ for all $a \in A$. Then $q := \theta(1_A)$ is a projection in $\mathcal{Q}_\omega$ which is nonzero as $\tau_{\mathcal{Q}_\omega}(q) = \gamma > 0$. We can view θ as a unital *-homomorphism taking values in $q\mathcal{Q}_\omega q$, which is isomorphic to $\mathcal{Q}_\omega$ by Proposition 1.3 (i). The uniqueness of the trace on $\mathcal{Q}_\omega$ ensures that $\tau_{q\mathcal{Q}_\omega q}(\theta(a)) = \frac{1}{\tau_{\mathcal{Q}_\omega}(q)} \tau_{\mathcal{Q}_\omega}(\theta(a)) = \tau_A(a)$ for $a \in A$, so (iib) holds.

Finally, note that if τ_A is faithful, then a unital *-homomorphism θ as in (iib) is injective, so A is quasidiagonal. $\square$

Remark 1.5. (i) When A is a non-unital C*-algebra one can define quasidiagonality in two natural equivalent ways: as in Definition 1.1 using c.p.c. maps in place of u.c.p. maps or by asking for the minimal unitisation $A^\sim$ to be quasidiagonal (see [11, Sections 2.2 and 7.1]).

(ii) Likewise, given a trace τ_A on a non-unital C*-algebra A, we obtain a trace $\tau_{A^\sim}$ on $A^\sim$ by the formula

$$(1.9) \qquad \tau_{A^\sim}(\lambda 1 + a) := \lambda + \tau_A(a), \quad \lambda \in \mathbb{C}, \ a \in A.$$

We define τ_A to be quasidiagonal if the trace $\tau_{A^\sim}$ is, or equivalently, if Definition 1.2 holds using c.p.c. maps in place of u.c.p. maps.

(iii) It is well known that if the trace τ_A is faithful on the non-unital C*-algebra A, then so too is $\tau_{A^\sim}$.[4] In particular, a faithful trace τ_A on A is quasidiagonal provided the faithful trace $\tau_{A^\sim}$ on $A^\sim$ is. In this case $A^\sim$, and hence also A, is quasidiagonal.

The next lemma provides the projections in $\mathcal{Q}_\omega$ used to rescale the trace in the "stable uniqueness across the interval" procedure of Section 5.

Lemma 1.6. *Let $k \in \mathcal{Q}_\omega$ be a positive contraction such that $\tau_{\mathcal{Q}_\omega}(h(k)) = \int_0^1 h(t)\,dt$ for all $h \in C([0,1])$. For each relatively open interval $I \subseteq [0,1]$, there exists a projection $q_I \in \mathcal{Q}_\omega$ such that the family $(q_I)_I$ satisfies:*

(i) q_I commutes with k, for all I;
(ii) q_I acts as a unit on $h(k)$, for all $h \in C_0(I)$ and all I;
(iii) if $h \in C([0,1])$ satisfies $h|_I \equiv 1$, then $h(k)q_I = q_I$;
(iv) if I, J are disjoint, then $q_I q_J = 0$;
(v) for all I, $\tau_{\mathcal{Q}_\omega}(q_I)$ is equal to the length, $|I|$, of I.

Proof. Represent k by a sequence $(k_n) \in \ell^\infty(\mathcal{Q})$ of positive contractions with discrete spectrum. Fix a relatively open interval I in $[0,1]$. For each n, let $q_{I,n} \in \mathcal{Q}$ denote the spectral projection of k_n corresponding to the interval I, and then set q_I to be the element of $\mathcal{Q}_\omega$ represented by $(q_{I,n})$.

As each $q_{I,n}$ commutes with k_n, it follows that q_I commutes with k, hence (i) holds. For $h \in C_0(I)$, $h(k)$ is represented by $(h(k_n))$. Since each $q_{I,n}$ acts as a unit on $h(k_n)$, it follows that q_I acts as a unit on $h(k)$, proving (ii). Likewise, in (iii), if $h \in C([0,1])$ satisfies $h|_I \equiv 1$, then $h(k_n)q_{I,n} = q_{I,n}$ for each n so that $h(k)q_I = q_I$. For (iv), note that if $I \cap J = \varnothing$, then $q_{I,n}q_{J,n} = 0$ for all n, whence $q_I q_J = 0$.

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[4]Every positive element of $A^\sim \backslash A$ is, up to scalar multiplication, of the form $1 - a$ where $a \in A$ is a self-adjoint whose positive part a_+ is a contraction. Since a_+ isn't the unit, $x(1 - a_+)x^* \neq 0$ for some $x \in A$, so that $\tau_A(x(1 - a_+)x^*) > 0$. Hence $\tau_A(a_+) < 1$ (see [66, Proposition 2.11], for example) so that $\tau_{A^\sim}(1 - a) \geq 1 - \tau_A(a_+) > 0$.

For (v), note that (ii) gives

$$(1.10) \qquad \tau_{\mathcal{Q}_\omega}(q_I) \geq \sup_{h \in C_0(I)_+^1} \int_0^1 h(t)dt = |I|.$$

Likewise, by (iii),

$$(1.11) \qquad \tau_{\mathcal{Q}_\omega}(q_I) \leq \inf_{h \in C([0,1])_+^1, \ h|_I \equiv 1} \int_0^1 h(t)dt = |I|.$$

These two inequalities yield (v). $\hfill\square$

Remark. The family of projections (q_I) in Lemma 1.6 are not canonical as they depend on the choice of lift for k in $\ell^\infty(\mathcal{Q})$.

We now turn to the universal coefficient theorem (UCT) which is a key component of the stable uniqueness theorem in Section 3. The definition, given below for completeness, is in terms of a natural sequence relating Kasparov's bivariant KK-groups to K-theory.

Definition 1.7 (cf. [60, Theorem 1.17]). A separable C*-algebra A is said to satisfy the *universal coefficient theorem* (UCT) if

$$0 \to \mathrm{Ext}(K_*(A), K_{*+1}(B)) \to$$
$$(1.12) \qquad KK(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

is an exact sequence, for every σ-unital C*-algebra B.

In [60], Rosenberg and Schochet showed that a large class of separable C*-algebras satisfy the UCT (the collection of C*-algebras satisfying the UCT is now known as the UCT class), and established closure properties; their work shows that the UCT class consists of precisely those separable C*-algebras which are KK-equivalent to abelian C*-algebras (this precise statement can be found as [3, Theorem 23.10.5]). In particular, note that if A satisfies the UCT, then so too does the unitisation $C_0((0,1), A)^\sim$ of the suspension of A.[5]

It was shown by Tu in [70, Lemma 3.5 and Proposition 10.7] that the C*-algebra of each amenable étale groupoid satisfies the UCT. In particular, this includes the group C*-algebras of countable discrete amenable groups.

We end this section by setting out our conventions regarding matrix ampflications which will appear frequently throughout the paper.

Notation 1.8. For a *-algebra A and $n \in \mathbb{N}$, we freely identify $A \otimes M_n$ and $M_n(A)$. Given n elements, $x_1, \ldots, x_n \in A$, we write $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ for the diagonal matrix in $M_n(A)$ with entries $x_1, x_2, \ldots, x_n$. Similar conventions apply to maps: given $\theta_1, \ldots, \theta_n : A \to B$, write $\theta_1 \oplus \cdots \oplus \theta_n : A \to M_n(B)$ for the map given by $a \mapsto \theta_1(a) \oplus \cdots \oplus \theta_n(a)$. When the elements involved are constant, we sometimes use $x^{\oplus n}$ to denote the diagonal element of $M_n(A)$ with constant entry x, and likewise, given $\theta : A \to B$, we write $\theta^{\oplus n}$ for the map $x \mapsto \theta(x)^{\oplus n}$.

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[5]If A is KK-equivalent to the abelian C*-algebra B, then $C_0((0,1), A)^\sim$ is KK-equivalent to the abelian C*-algebra $C_0((0,1), B)^\sim$ (for example, by [3, 19.1.2 (c) and (d)]).

2. Two Lebesgue trace cones

The central purpose of this section is to produce the two Lebesgue trace cones over A in $\mathcal{Q}_\omega$ (obtained in Lemma 2.6 below) with unital sum, which are the inputs into the "stable uniqueness across the interval" procedure. We start by using Cuntz subequivalence, originating in [18] and further developed in [57, 17] (we refer to [1] for a full account of the Cuntz semigroup), and strict comparison of positive elements (which has its origins in [2]) to record two technical observations.

Recall that for $a, b \in A_+$, a is said to be *Cuntz below* b if there exists a sequence (x_n) of elements of A such that $a = \lim_n x_n b x_n^*$; a is called *Cuntz equivalent* to b if a is Cuntz below b and b is Cuntz below a. For $\varepsilon > 0$, $(a - \varepsilon)_+$ denotes the functional calculus output given by applying the function $h(t) = \max\{t - \varepsilon, 0\}$ to a. With this notation, a is Cuntz below b in A if and only if for all $\varepsilon > 0$, there exists $v \in A$ such that $(a - \varepsilon)_+ = vbv^*$ (see [57, Proposition 2.4], which shows that a is Cuntz below b if and only if for all $\varepsilon > 0$, there exist $\delta > 0$ and $w \in A$ such that $(a - \varepsilon)_+ = w(b - \delta)_+ w^*$).

A C*-algebra A is said to have *strict comparison of positive elements with respect to bounded traces*, if whenever $k \in \mathbb{N}$ and $a, b \in (A \otimes M_k)_+$ satisfy

$$(2.1) \qquad d_{\tau \otimes \mathrm{Tr}_k}(a) < d_{\tau \otimes \mathrm{Tr}_k}(b)$$

for all $\tau \in T(A)$ (where $d_\tau(a) := \lim_{n \to \infty} \tau(a^{1/n})$), it follows that a is Cuntz below b in $A \otimes M_k$.[6] The functions $d_{\tau \otimes \mathrm{Tr}_k}$ above provide functionals on the Cuntz semigroup ([31, Proposition 4.2]); in particular

$$(2.2) \quad d_{\tau \otimes \mathrm{Tr}_k}(a) + d_{\tau \otimes \mathrm{Tr}_k}(b) = d_{\tau \otimes \mathrm{Tr}_k}(a + b), \quad a, b \in (A \otimes M_k)_+, \ ab = 0;$$

$$(2.3) \qquad d_{\tau \otimes \mathrm{Tr}_k}(a) \leq d_{\tau \otimes \mathrm{Tr}_k}(b), \quad a \text{ is Cuntz below } b \text{ in } A \otimes M_k.$$

The key example for us is that $\mathcal{Q}_\omega$ has strict comparison with respect to its unique bounded trace. This follows from the corresponding fact for $\mathcal{Q}$ (by [57, Theorem 5.2 (a)])[7] and [6, Lemma 1.23], for example, which shows that this property passes to ultraproducts.

Our first use of Cuntz comparison is the following, now-standard consequence of Ciuperca and Elliott's classification of *-homomorphisms from $C_0((0, 1])$ to stable rank one C*-algebras by their Cuntz semigroup data, from [15, Theorem 4] (and extended in [56] and [36]). We shall use this result in Lemma 2.6 to obtain our second Lebesgue trace cone as a unitary conjugate of the first cone. The argument below is very similar to the proof of [62, Lemma 5.1].

Lemma 2.1. *(i) Let A be a unital C*-algebra of stable rank one and strict comparison of positive elements with respect to bounded traces. Suppose that $a_1, a_2 \in A$ are positive contractions such that*

$$(2.4) \qquad \tau(h(a_1)) = \tau(h(a_2)) > 0, \quad \tau \in T(A), \ h \in C_0((0, 1])_+ \backslash \{0\}.$$

Then a_1 and a_2 are approximately unitarily equivalent.

---

[6]Strict comparison is usually defined using densely defined lower semicontinuous 2-quasitraces, but we do not need this version of the definition.

[7]Up to scalar multiplication the only lower semicontinuous dimension function on $\mathcal{Q}$ arises from the unique trace.

(ii) Suppose that $a_1, a_2 \in \mathcal{Q}_\omega$ are positive contractions such that

$$(2.5) \qquad \tau_{\mathcal{Q}_\omega}(h(a_1)) = \tau_{\mathcal{Q}_\omega}(h(a_2)) > 0, \quad h \in C_0((0,1])_+ \backslash \{0\}.$$

Then a_1 and a_2 are unitarily equivalent.

Proof. (i) By [15, Theorem 4], it suffices to show that $h(a_1)$ is Cuntz equivalent to $h(a_2)$ for all nonzero $h \in C_0((0,1])_+$. For such h, let $\varepsilon > 0$, and define

$$(2.6) \qquad U := \{ t \in (0,1] \mid 0 < h(t) < \varepsilon \}.$$

This open set is nonempty and therefore there exists a nonzero function $g \in C_0(U)_+$ of norm one. Then for any trace $\tau \in T(A)$, we have $d_\tau(g(a_1)) \geq \tau(g(a_1)) > 0$ by (2.4). Since $(h(a_1) - \varepsilon)_+$ is orthogonal to $g(a_1)$,

$$
\begin{aligned}
d_\tau((h(a_1) - \varepsilon)_+) \;&<\; d_\tau((h(a_1) - \varepsilon)_+) + d_\tau(g(a_1)) \\
&\overset{(2.2)}{=} d_\tau((h(a_1) - \varepsilon)_+ + g(a_1)) \\
&\overset{(2.3)}{\leq} d_\tau(h(a_1)) \\
(2.7) \qquad &\overset{(2.4)}{=} d_\tau(h(a_2)), \quad \tau \in T(A)
\end{aligned}
$$

as $h((a_1) - \varepsilon)_+ + g(a_1)$ is Cuntz below $h(a_1)$. Since A has strict comparison of positive elements by bounded traces, it follows that $(h(a_1) - \varepsilon)_+$ is Cuntz below $h(a_2)$; as ε is arbitrary, $h(a_1)$ is Cuntz below $h(a_2)$. Symmetrically, we also obtain that $h(a_2)$ is Cuntz below $h(a_1)$, as required.

(ii) Since stable rank one passes to ultrapowers (see [62, Lemma 2.4], a simple modification of [47, Lemma 19.2.2 (1)] to ultrapowers), $\mathcal{Q}_\omega$ has stable rank one and strict comparison with respect to its unique trace $\tau_{\mathcal{Q}_\omega}$. The result now follows from (i), noting that a standard reindexing argument shows that two positive elements in an ultrapower are unitarily equivalent if and only if they are approximately unitarily equivalent (see [6, Lemma 1.17 (i)], for example). □

Our second application of strict comparison is a standard computation familiar to experts; it will be used in the proof of Theorem A in order to verify Δ-fullness as in Definition 3.4.

Lemma 2.2. *Let A be a unital C^*-algebra with strict comparison of positive elements by bounded traces, such as $\mathcal{Q}_\omega$. Let $a \in A_+^1$. If $m \in \mathbb{N}$ satisfies $\tau(a) > \frac{2}{m}$ for all $\tau \in T(A)$ then there exist m^2 contractions $v_1, \ldots, v_{m^2} \in A$ such that*

$$(2.8) \qquad 1_A = \sum_{i=1}^{m^2} v_i a v_i^*.$$

Proof. Set $\delta := \frac{1}{2} \min_{\tau \in T(A)} \tau(a)$ so that $m\delta > 1$.[8] For $\tau \in T(A)$, we have

$$(2.9) \qquad d_\tau((a - \delta)_+^2) \geq \tau((a - \delta)_+) \geq \tau(a) - \delta \geq \delta,$$

so that

$$(2.10) \qquad d_{\tau \otimes \mathrm{Tr}_m}(((a-\delta)^2)^{\oplus m}) > d_{\tau \otimes \mathrm{Tr}_m}(1_A \oplus 0_A^{\oplus (m-1)}).$$

---

[8]This minimum exists as $T(A)$ is compact.

Strict comparison of positive elements shows that $1_A \oplus 0_A^{\oplus(m-1)}$ is Cuntz below $((a-\delta)^2)^{\oplus m}$ in $A \otimes M_m$ so that (as $(1_A - \varepsilon)_+ = (1-\varepsilon)1_A$), there exist $b_1, \ldots, b_m \in A$ such that

$$(2.11) \qquad 1_A = \sum_{j=1}^{m} b_j (a-\delta)_+^2 b_j^*,$$

by [57, Proposition 2.4], as discussed after the definition of Cuntz comparison above. In particular, $\|b_j(a-\delta)_+\| \le 1$ for each j.

Define $h \in C([0,1])$ by

$$(2.12) \qquad h(t) := \begin{cases} \frac{1}{\sqrt{t}}, & t \ge \delta; \\ \frac{t}{\delta^{3/2}}, & t \in [0, \delta]. \end{cases}$$

Note that $h(t)^2 t = 1$ for $t \ge \delta$, so $(a-\delta)_+ h(a)^2 a = (a-\delta)_+$. Writing $c_j := m^{-1/2} b_j (a-\delta)_+ h(a)$, which is a contraction as $\|c_j\| \le m^{-1/2}\|h(a)\| \le (m\delta)^{-1/2} < 1$, we have

$$1_A = \sum_{j=1}^{m} b_j(a-\delta)_+^2 b_j^*$$

$$(2.13) \qquad\qquad = m\sum_{j=1}^{m} c_j a c_j^*,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Our two Lebesgue trace cones are obtained from the tracially large order zero maps constructed in [62, Proposition 3.2] using Connes' uniqueness of the injective II$_1$ factor ([16]). The form we need is given in Proposition 2.4 below.

Definition 2.3 ([78, Definition 1.3]). Let A, B be C*-algebras. A completely positive map $\phi : A \to B$ is said to be *order zero* if for every $a, b \in A_+$ with $ab = 0$, one has $\phi(a)\phi(b) = 0$.

Since traces on a unital C*-algebra A form a weak*-compact convex set, $T(A)$ is the weak*-closed convex hull of extremal traces on A by the Krein–Milman theorem. Recall that $\tau \in T(A)$ is extremal if and only if the associated GNS representation $\pi_\tau(A)''$ is a factor ([24, Theorem 6.7.3]).

Proposition 2.4. *Let A be a separable, unital and nuclear C*-algebra and let τ_A be a trace on A. Then there exists a c.p.c. order zero map $\Omega : A \to \mathcal{Q}_\omega$ such that*

$$(2.14) \qquad \tau_{\mathcal{Q}_\omega}(\Omega(a)\Omega(1_A)^{n-1}) = \tau_A(a), \quad a \in A, \ n \in \mathbb{N}.$$

Proof. The proof of [62, Proposition 3.2], establishes the proposition in the case that τ_A is an extremal trace on A for which the GNS representation $\pi_{\tau_A}(A)''$ is type II$_1$ (and hence a copy of the hyperfinite II$_1$ factor) except that (2.14) is stated by the conditions that $\tau_{\mathcal{Q}_\omega} \circ \Omega = \tau_A$ and $1_{\mathcal{Q}_\omega} - \Omega(1_A)$ is in the kernel associated to $\tau_{\mathcal{Q}_\omega}$.[9] To obtain (2.14) for $n > 1$ from these

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[9]The key point is that the argument in [62, Proposition 3.2] is readily seen to be valid when $\pi_{\tau_A}(A)''$ is a copy of the hyperfinite II$_1$ factor; this is the only use of the hypothesis in the statement of [62, Proposition 3.2] that A has no finite dimensional quotients.

conditions, by the Cauchy–Schwarz inequality we have

$$|\tau_{\mathcal{Q}_\omega}(y(1_{\mathcal{Q}_\omega} - \Omega(1_A)))| \leq \tau_{\mathcal{Q}_\omega}(y^*y)^{1/2}\tau_{\mathcal{Q}_\omega}((1_{\mathcal{Q}_\omega} - \Omega(1_A))^2)^{1/2}$$
$$(2.15) \qquad\qquad\qquad = 0, \quad y \in \mathcal{Q}_\omega,$$

as $1_{\mathcal{Q}_\omega} - \Omega(1_A)$ lies in the kernel associated $\tau_{\mathcal{Q}_\omega}$. Setting $y := \Omega(a)\Omega(1_A)^{n-2}$ allows us to prove (2.14) inductively.

In contrast, if τ_A is a extremal trace on A such that $\pi_{\tau_A}(A)''$ is a type I factor, we may define Ω to be the composition of the unital embedding $\pi_{\tau_A} : A \to \pi_{\tau_A}(A)''$ with some unital (and necessarily trace preserving) embedding $\pi_{\tau_A}(A)'' \hookrightarrow \mathcal{Q}_\omega$. This is a *-homomorphism with $\tau_{\mathcal{Q}_\omega} \circ \Omega = \tau_A$. Thus the proposition holds when τ_A is an extremal trace on A.

The case of a general trace τ_A on A follows by approximating by convex combinations and a standard reindexing argument, as we now explain. The reindexing argument we use is Kirchberg's ε-test from [40, Lemma A.1] or [41, Lemma 3.1]. For $i \in \mathbb{N}$, write X_i for the collection of *-linear maps $A \to \mathcal{Q}$. Then, as set out in [6, Lemma 1.12], there exists a countable collection of functions $f_i^{(k)} : X_i \to [0, \infty]$ indexed by $k, i \in \mathbb{N}$, such that a sequence $(\phi_i)_{i=1}^\infty$ in $\prod_{i=1}^\infty X_i$ (the set product) induces a c.p.c. order zero map $A \to \mathcal{Q}_\omega$ if and only if $\lim_{i\to\omega} f_i^{(k)}(\phi_i) = 0$ for all $k \in \mathbb{N}$. Fixing a countable dense subset $(a_k)_{k=1}^\infty$ in A, define functions $g_i^{(k,n)} : X_i \to [0, \infty]$ for $i, k, n \in \mathbb{N}$ by

$$(2.16) \qquad g_i^{(k,n)}(\phi_i) := |\tau_{\mathcal{Q}}(\phi_i(a_k)\phi_i(1_A)^{(n-1)}) - \tau_A(a_k)|.$$

In this way, a sequence $(\phi_i)_{i=1}^\infty$ in $\prod_{i=1}^\infty X_n$ induces a c.p.c. order zero map $A \to \mathcal{Q}_\omega$ satisfying (2.14) if and only if

$$(2.17) \qquad \lim_{i\to\omega} f_i^{(k)}(\phi_i) = \lim_{i\to\omega} g_i^{(k,n)}(\phi_i) = 0,$$

for all $k, n \in \mathbb{N}$.

Now fix $k \in \mathbb{N}$ and $\varepsilon > 0$. Find $m \in \mathbb{N}$, positive numbers $\lambda_1, \ldots, \lambda_m$ with $\sum_{j=1}^m \lambda_j = 1$, and extremal traces $\tau_1, \ldots, \tau_m$ on A such that

$$(2.18) \qquad \left|\sum_{j=1}^m \lambda_j \tau_j(a_r) - \tau_A(a_r)\right| < \varepsilon, \quad r = 1, \ldots, k.$$

Choose pairwise orthogonal projections $p_1, \ldots, p_m \in \mathcal{Q}_\omega$ which sum to $1_{\mathcal{Q}_\omega}$ and satisfy $\tau_{\mathcal{Q}_\omega}(p_j) = \lambda_j$ for each j. Then $p_j\mathcal{Q}_\omega p_j \cong \mathcal{Q}_\omega$ by Proposition 1.3 (i). By the first three paragraphs of this proof, for each j we can find a c.p.c. order zero map $\Omega_j : A \to p_j\mathcal{Q}_\omega p_j$ such that

$$(2.19) \qquad \tau_{p_j\mathcal{Q}_\omega p_j}(\Omega_j(a)\Omega_j(1_A)^{n-1}) = \tau_j(a), \quad a \in A, \ n \in \mathbb{N}.$$

Define $\Omega : A \to \mathcal{Q}_\omega$ by $\Omega(a) = \sum_{j=1}^m \Omega_j(a)$ for $a \in A$. Since the p_j are pairwise orthogonal, Ω is a c.p.c. order zero map, and it satisfies

$$\tau_{\mathcal{Q}_\omega}(\Omega(a)\Omega(1_A)^{n-1}) = \sum_{j=1}^m \lambda_j \tau_{p_j\mathcal{Q}_\omega p_j}(\Omega_j(a)\Omega_j(1_A)^{n-1})$$
$$(2.20) \qquad\qquad\qquad = \sum_{j=1}^m \lambda_j \tau_j(a), \quad a \in A, \ n \in \mathbb{N}.$$

Combining this with (2.18), it follows that any lift of Ω to a sequence $(\phi_i)_{i=1}^\infty$ in $\prod_{i=1}^\infty X_n$ has $\lim_{i\to\omega} g_i^{(r,n)}(\phi_i) \le \varepsilon$ for all $n \in \mathbb{N}$ and $r = 1,\dots,k$ and $\lim_{i\to\omega} f_i^{(l)}(\phi_i) = 0$ for all l. Thus by Kirchberg's ε-test, there exists a sequence $(\phi_i)_{i=1}^\infty$ in $\prod_i X_i$ satisfying (2.17), and hence providing the required c.p.c. order zero map $\Omega : A \to \mathcal{Q}_\omega$ satisfying (2.14). $\qquad\square$

Our two Lebesgue trace cones over A will be constructed so that their scalar parts agree with a *-homomorphism on $C([0,1])$. We encapsulate this property in the following definition.

Definition 2.5. Let A and E be unital C*-algebras and let $I \subset [0,1]$ be an interval. Let $\theta : C([0,1]) \to E$ be a unital *-homomorphism. We say that a *-homomorphism $\nu : C_0(I,A) \to E$ is *compatible* with θ if

$$(2.21) \qquad \nu(hx) = \theta(h)\nu(x) = \nu(x)\theta(h), \quad h \in C([0,1]), \; x \in C_0(I,A).$$

The essential feature of order zero maps we need in the next proposition is their correspondence with *-homomorphisms from cones as in [78, Corollary 3.1]: a c.p.c. map $\Omega : A \to B$ is order zero if and only if there exists a *-homomorphism $\pi_\Omega : C_0((0,1]) \otimes A \to B$ such that,

$$(2.22) \qquad \Omega(a) = \pi_\Omega(\mathrm{id}_{[0,1]} \otimes a), \quad a \in A.^{10}$$

Given a bounded interval $I \subset \mathbb{R}$, we write τ_{Leb} for the tracial functional on $C_0(I)$ induced by Lebesgue measure on I, i.e.

$$(2.23) \qquad \tau_{\mathrm{Leb}}(h) := \int_I h(t)\,dt, \quad h \in C_0(I).$$

This is a trace when I has length 1 (in particular for $I = [0,1]$, $(0,1]$ or $[0,1)$). A positive contraction a in a unital C*-algebra A is said to have Lebesgue spectral measure with respect to $\tau_A \in T(A)$ if

$$(2.24) \qquad \tau_A(h(a)) = \tau_{\mathrm{Leb}}(h), \quad h \in C([0,1]).$$

Note that a has Lebesgue spectral measure with respect to τ_A if and only if $1_A - a$ does.

Lemma 2.6. *Let A be a separable, unital and nuclear C*-algebra and $\tau_A \in T(A)$. Then there are *-homomorphisms*

$$(2.25) \quad \Theta : C([0,1]) \to \mathcal{Q}_\omega, \; \grave{\Phi} : C_0([0,1),A) \to \mathcal{Q}_\omega, \; \acute{\Phi} : C_0((0,1],A) \to \mathcal{Q}_\omega$$

such that:

  (i) Θ is unital and $\grave{\Phi}, \acute{\Phi}$ are compatible with Θ;
  (ii) $\tau_{\mathcal{Q}_\omega} \circ \grave{\Phi} = \tau_{\mathrm{Leb}} \otimes \tau_A$;
  (iii) $\tau_{\mathcal{Q}_\omega} \circ \acute{\Phi} = \tau_{\mathrm{Leb}} \otimes \tau_A$.

Proof. By Proposition 2.4, there exists a c.p.c. order zero map $\Omega : A \to \mathcal{Q}_\omega$ satisfying (2.14). Let $\pi_\Omega : C_0((0,1],A) \to \mathcal{Q}_\omega$ be the *-homomorphism associated to Ω satisfying (2.22). Let $k \in \mathcal{Q}_+$ be a positive contraction with spectrum $[0,1]$ such that k has Lebesgue spectral measure with respect to $\tau_{\mathcal{Q}}$. Let $\pi_k : C_0((0,1]) \to \mathcal{Q}$ denote the *-homomorphism defined by $\pi_k(h) := h(k)$.

---

[10]We use $\mathrm{id}_{[0,1]}$ for the identity function on $[0,1]$ satisfying $\mathrm{id}_{[0,1]}(t) = t$. This should not be confused with $1_{C([0,1])}$, the unit of $C([0,1])$.

Define $\alpha : (0,1] \times (0,1] \to (0,1]$ by $\alpha(s,t) := st$ and use this to define $\acute{\Phi} : C_0((0,1], A) \to \mathcal{Q} \otimes \mathcal{Q}_\omega$ by

$$(2.26) \qquad \acute{\Phi}(x) := (\pi_k \otimes \pi_\Omega)(x \circ \alpha), \quad x \in C_0((0,1], A).^{11}$$

An isomorphism $\mathcal{Q} \otimes \mathcal{Q} \cong \mathcal{Q}$ can be used to induce a unital embedding $\mathcal{Q} \otimes \mathcal{Q}_\omega \to \mathcal{Q}_\omega$. Hence we can regard $\acute{\Phi}$ as taking values in $\mathcal{Q}_\omega$. Define $\Theta : C([0,1]) \to \mathcal{Q}_\omega$ to be the unitisation of $\acute{\Phi}|_{C_0((0,1], \mathbb{C}1_A)}$. In this way $\acute{\Phi}$ is certainly compatible with Θ.

For $a \in A$ and $n \in \mathbb{N}$, the definition of $\acute{\Phi}$ ensures that

$$\acute{\Phi}(\mathrm{id}_{[0,1]}^n \otimes a) = (\pi_k \otimes \pi_\Omega)(\mathrm{id}_{[0,1]}^n \otimes \mathrm{id}_{[0,1]}^n \otimes a)$$

$$(2.27) \qquad\qquad \overset{(2.22)}{=} k^n \otimes \left(\Omega(a)\Omega(1_A)^{n-1} \right).$$

Thus,

$$\tau_{\mathcal{Q}_\omega}(\acute{\Phi}(\mathrm{id}_{[0,1]}^n \otimes a)) \overset{(2.27)}{=} \tau_{\mathcal{Q}}(k^n)\tau_{\mathcal{Q}_\omega}(\Omega(a)\Omega(1_A)^{n-1})$$

$$(2.28) \qquad\qquad \overset{(2.14),(2.24)}{=} \tau_{\mathrm{Leb}}(\mathrm{id}_{[0,1]}^n)\tau_A(a).$$

By linearity and density,

$$(2.29) \qquad\qquad \tau_{\mathcal{Q}_\omega} \circ \acute{\Phi} = \tau_{\mathrm{Leb}} \otimes \tau_A$$

proving (iii).

Since the positive contraction $\Theta(\mathrm{id}_{[0,1]}) = \acute{\Phi}(\mathrm{id}_{[0,1]} \otimes 1_A)$ has Lebesgue spectral measure with respect to $\tau_{\mathcal{Q}_\omega}$, so too does $1_{\mathcal{Q}_\omega} - \Theta(\mathrm{id}_{[0,1]})$. Therefore, by Lemma 2.1 (ii), there is a unitary $u \in \mathcal{Q}_\omega$ such that

$$(2.30) \qquad \Theta(1_{C([0,1])} - \mathrm{id}_{[0,1]}) = 1_{\mathcal{Q}_\omega} - \acute{\Phi}(\mathrm{id}_{[0,1]}) = u\acute{\Phi}(\mathrm{id}_{[0,1]})u^*.$$

Define $\grave{\Phi}$ to be the composition

$$(2.31) \qquad \grave{\Phi} : C_0([0,1), A) \overset{\sigma}{\longrightarrow} C_0((0,1], A) \overset{\acute{\Phi}}{\longrightarrow} \mathcal{Q}_\omega \overset{\mathrm{Ad}(u)}{\longrightarrow} \mathcal{Q}_\omega,$$

where σ is the flip map given by $\sigma(x)(t) := x(1-t)$ for $x \in C_0([0,1), A)$, $t \in (0,1]$. By construction, $\sigma(1_{C([0,1])} - \mathrm{id}_{[0,1]}) = \mathrm{id}_{[0,1]}$, and hence

$$\grave{\Phi}(1_{C([0,1])} - \mathrm{id}_{[0,1]}) = u\acute{\Phi}(\mathrm{id}_{[0,1]})u^*$$

$$(2.32) \qquad\qquad \overset{(2.30)}{=} \Theta(1_{C([0,1])} - \mathrm{id}_{[0,1]}).$$

Since Θ is unital and the cone $C_0([0,1))$ is generated by $1_{C([0,1])} - \mathrm{id}_{[0,1]}$, it follows that

$$(2.33) \qquad\qquad \grave{\Phi}|_{C_0([0,1), \mathbb{C}1_A)} = \Theta|_{C_0([0,1))},$$

and so $\grave{\Phi}$ is compatible with Θ, establishing (i).

---

[11]Alternatively $\acute{\Phi}$ can be defined as the *-homomorphism $C_0((0,1], A) \to \mathcal{Q} \otimes \mathcal{Q}_\omega$ associated by (2.22) to the c.p.c. order zero map $A \to \mathcal{Q} \otimes \mathcal{Q}_\omega$ given by $a \mapsto k \otimes \Omega(a)$.

For (ii), we compute

$$
\begin{aligned}
\tau_{\mathcal{Q}_\omega} \circ \grave{\Phi} \;\; &\overset{(2.31)}{=} \;\; \tau_{\mathcal{Q}_\omega} \circ \mathrm{Ad}(u) \circ \acute{\Phi} \circ \sigma \\
&= \;\; \tau_{\mathcal{Q}_\omega} \circ \acute{\Phi} \circ \sigma \\
&\overset{(2.29)}{=} \;\; (\tau_{\mathrm{Leb}} \otimes \tau_A) \circ \sigma \\
&= \;\; \tau_{\mathrm{Leb}} \otimes \tau_A,
\end{aligned}
$$

(2.34)

as integration with respect to Lebesgue measure on $[0,1]$ is certainly invariant under flipping the interval. □

3. A controlled stable uniqueness theorem

This section contains a small modification of a stable uniqueness theorem from [21] (which was in turn inspired by Lin's paper [43]). A similar modification was given by Lin in [45, Theorem 5.9]; we have chosen to give an argument (based on [21]) to make it transparent how the UCT hypothesis is used in Theorem A. Recall that a *-homomorphism $\iota : A \to B$ is said to be *totally full*[12] if for every nonzero $a \in A$, $\iota(a)$ is full in B, i.e. $\overline{\mathrm{span} B \iota(a) B} = B$. With this, our starting point is the stable uniqueness theorem below, from [21]. For a separable C*-algebra A and a σ-unital C*-algebra B, the group $KK_{\mathrm{nuc}}(A,B)$ is Skandalis' modification of $KK(A,B)$ obtained from the Cuntz picture by working only with strictly nuclear maps and homotopies ([64, Section 2]) and, as such, when A or B is nuclear, the canonical homomorphism $KK_{\mathrm{nuc}}(A,B) \to K(A,B)$ is an isomorphism. This also holds when A is KK-equivalent to a nuclear C*-algebra ([64, Propositions 3.2 and 3.3]), and hence whenever A satisfies the UCT.

Theorem 3.1 (Dadarlat–Eilers; cf. [21, Theorem 4.5])**.** *Let A, B be unital C*-algebras with A separable. Let $\iota : A \to B$ be a totally full unital *-homomorphism, and suppose that $\phi, \psi : A \to B$ are nuclear *-homomorphisms inducing the same class in $KK_{\mathrm{nuc}}(A,B)$ and such that $\phi(1_A)$ is unitarily equivalent to $\psi(1_A)$. Then, for any finite subset $\mathcal{G} \subset A$ and $\delta > 0$, there exist $n \in \mathbb{N}$ and a unitary $u \in M_{n+1}(B)$ such that*

$$
(3.1) \qquad \| u(\phi(a) \oplus \iota^{\oplus n}(a))u^* - (\psi(a) \oplus \iota^{\oplus n}(a)) \| < \delta, \quad a \in \mathcal{G}.
$$

Proof. This is a special case of [21, Theorem 4.5]. The totally full map ι induces a representation $\gamma : A \to M(\mathcal{K} \otimes B)$ by $\gamma(a) = \iota^{\oplus \infty}(a)$, for $a \in A$, which is nuclearly absorbing in the sense of [21, Definition 2.4] by [21, Theorem 2.22]. The representation γ is quasidiagonal[13] in the sense of [21, Definition 2.5] with a quasidiagonalisation of the form $\gamma_n = \iota^{\oplus n} : A \to M_n(B)$, and so we are in a situation covered by [21, Theorem 4.5], yielding the result. □

---

[12]Unital totally full *-homomorphisms are called unital full embeddings in [21].

[13]Here the term quasidiagonal representation is used in the setting of representations on Hilbert modules; it differs from the use of the term quasidiagonality elsewhere in this paper. The quasidiagonality of γ is witnessed by the projections $e_n \otimes 1_B \in \mathcal{K} \otimes B$ which commute exactly with $\gamma(A)$, where (e_n) is an increasing sequence of projections with e_n of rank n, converging strictly to the identity operator. Thus, identifying $e_n(M(\mathcal{K} \otimes B))e_n$ with $M_n(B)$, one has $\gamma_n(a) = e_n(\gamma(a))e_n = \iota^{\oplus n}(a)$ for $a \in A$.

In the above stable uniqueness theorem, the n depends not only on $\mathcal{G}$ and δ, but also on the maps ι, ϕ and ψ. From this, Dadarlat and Eilers obtain a stable uniqueness theorem for simple domains A ([21, Theorem 4.12]), in which n depends only on $\mathcal{G}$ and δ and not on ι, ϕ and ψ by using a sequence of counterexamples and Theorem 3.1. For completeness, and in order to show where the UCT hypothesis is used in our main theorem, we give an account of this argument below, which is the basis for the stable uniqueness theorem we require. Firstly, in order to control the KK-class of a *-homomorphism arising from a sequence of maps, Dadarlat and Eilers impose conditions on the behaviour of the allowed target C*-algebras.

Definition 3.2 ([21, Definition 4.9]). A C*-algebra B is an *admissible target algebra of finite type*[14] if B is unital, B has real rank zero, and the following conditions hold:

(i) for $k \in \mathbb{N}$ and projections $p, q \in B \otimes M_k$, if $[p] = [q]$ in $K_0(B)$ then $p \oplus 1_B$ is Murray–von Neumann equivalent to $q \oplus 1_B$;
(ii) the canonical map from unitaries in B to $K_1(B)$ is surjective;
(iii) for any $r \in K_0(B)$, if $nr \geq 0$ for some $n \in \mathbb{N}$, then $[1_B] + r \geq 0$;
(iv) for any $r \in K_0(B)$ and any $n \in \mathbb{N}$, there is $s \in K_0(B)$ such that $-[1_B] \leq s \leq [1_B]$ and $r - s \in nK_0(B)$.

In the proof of Theorem A we only require that $\mathcal{Q}_\omega$ is an admissible target algebra of finite type. To see this, first note that basic properties of $\mathcal{Q}$ show that $\mathcal{Q}$ is an admissible target algebra of finite type.[15] Given a sequence (B_n) of admissible target algebras of finite type, the product $\prod_n B_n$ and sequence algebra $\prod_n B_n / \sum_n B_n$ are both admissible target algebras of finite type by [21, Theorem 4.10 (iv)]. This is readily adapted to ultraproducts in place of $\prod_n B_n / \sum_n B_n$, so that $\mathcal{Q}_\omega$ is an admissible target algebra of finite type.

The second ingredient in [21, Theorem 4.12] is the *total K-theory* of a C*-algebra A, $\underline{K}(A)$, defined as the direct sum of the K-theory group, $K_*(A)$, and all K-theory groups with coefficients, $K_*(A; \mathbb{Z}/n\mathbb{Z})$, $n \in \mathbb{N}$, from [63] (see Theorem 6.4 there in particular); in the literature, $\underline{K}(A)$ is sometimes just called the K-theory of A with coefficients. As noted in [23, Section 4], there exists a C*-algebra C such that $\underline{K}(A) \cong K_0(A \otimes C)$ for every C*-algebra A. In this way $\underline{K}(\cdot)$ is a functor from C*-algebras to abelian groups invariant under homotopy and stable isomorphism such that, for *-homomorphisms $\phi, \psi : A \to B$, one has (identifying $\underline{K}(M_2(B))$ with $\underline{K}(B)$)

$$(3.2) \qquad (\phi \oplus \psi)_* = \phi_* + \psi_* : \underline{K}(A) \to \underline{K}(B).$$

When considering maps between total K-theory groups, one considers only those morphisms respecting certain additional structure (see [21, Section 3.1] or [22]); the set of these morphisms is denoted by $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(A))$. The essential use of the UCT in [21, Theorem 4.12] is to take sequences of maps from a separable C*-algebra into admissible target algebras of finite

---

[14][21] also considers admissible target algebras of infinite type; we do not need these here.

[15]Indeed in (i) $[p] = [q]$ in $K_0(\mathcal{Q})$ implies p is Murray–von Neumann equivalent to q; in (ii) $K_1(\mathcal{Q}) = 0$, so surjectivity is automatic; in (iii) $K_0(\mathcal{Q})$ is unperforated — if $nr \geq 0$ in $K_0(\mathcal{Q})$ for some $n \in \mathbb{N}$, then $r \geq 0$; finally in (iv), for each n, we have $K_0(\mathcal{Q}) = nK_0(\mathcal{Q})$, so one can always take $s = 0$.

type which agree on total K-theory, and obtain induced maps which, when the domain satisfies the UCT, give the same KK-class; this is the content of the next proposition, which is the only point that the UCT enters into the proof of Theorem A.

Proposition 3.3 (Dadarlat–Eilers [21])**.** *Let A be a separable* C*-*algebra that satisfies the UCT and, for each $n \in \mathbb{N}$, let B_n be an admissible target algebra of finite type. Set*

$$(3.3) \qquad B_\infty := \prod_{n=1}^\infty B_n / \{(b_n)_{n=1}^\infty \mid \lim_n \|b_n\| = 0\}.$$

*If $\phi_n, \psi_n : A \to B_n$ are *-homomorphisms satisfying $(\phi_n)_* = (\psi_n)_* : \underline{K}(A) \to \underline{K}(B_n)$ for each n, then the *-homomorphisms $A \to B_\infty$ induced by the sequences $(\phi_n)_n$ and $(\psi_n)_n$ have the same class in $KK_{\mathrm{nuc}}(A, B_\infty)$.*

Proof. Denote the two *-homomorphisms in question by Φ and Ψ respectively. By [21, Theorem 4.10 (ii)], both $(\phi_n)_n$ and $(\psi_n)_n$ induce the same map $\underline{K}(A) \to \underline{K}(\prod_n B_n)$, whence $\Phi_* = \Psi_*$ in $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B_\infty))$. As A satisfies the UCT, [21, Theorem 4.10 (iii)] shows that the natural map

$$(3.4) \qquad KK(A, B_\infty) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B_\infty))$$

is an isomorphism, so that $[\Phi] = [\Psi]$ in $KK(A, B_\infty)$. Since A is separable and satisfies the UCT, $KK(A, B_\infty) = KK_{\mathrm{nuc}}(A, B_\infty)$ by [64] (see the discussion at the beginning of the section), so $[\Phi] = [\Psi]$ in $KK_{\mathrm{nuc}}(A, B_\infty)$. $\square$

With the above ingredients in place, the other detail needed to prove [21, Theorem 4.12] is a device for ensuring that given a sequence of totally full maps $\iota_n : A \to B_n$ into admissible target algebras of finite type, the induced map $I : A \to B_\infty$ is totally full. In general this is false, so in [21, Theorem 4.12] Dadarlat and Eilers consider simple unital domain algebras A, as in this case, every unital *-homomorphism is totally full. When we apply the stable uniqueness theorem in the proof of Theorem A, the domain will be the nonsimple C*-algebra $C_0((0,1), A)^\sim$ for some separable unital and nuclear C*-algebra A satisfying the UCT. To ensure that the resulting I is totally full we use the following notion of controlled fullness.

Definition 3.4. Let B, C be unital C*-algebras. A *control function* on C is a function $\Delta : C_+^1 \setminus \{0\} \to \mathbb{N}$. Say that a unital *-homomorphism $\iota : C \to B$ is Δ-*full* if for every nonzero $x \in C_+^1$ there exist contractions $b_1, \dots, b_{\Delta(x)} \in B$ such that

$$(3.5) \qquad 1_B = \sum_{i=1}^{\Delta(x)} b_i^* \iota(x) b_i.$$

Note that a *-homomorphism $\iota : C \to B$ is totally full if and only if there exists a control function $\Delta : C_+^1 \setminus \{0\} \to \mathbb{N}$ such that ι is Δ-full.

Using these control functions we obtain the small generalisation[16] of [21, Theorem 4.12], with constants independent of the exact form of the maps ι, ϕ

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[16]If C is simple and unital, then there exists a control function Δ_C such that every unital *-homomorphism $C \to B$ is Δ_C-full; in this way Theorem 3.5 does generalise [21, Theorem 4.12].

and ψ, and depending only on how full the map ι is in terms of these control functions (and on the finite set and tolerance). The proof is essentially the same as that in [21]. A similar adaptation of the stable uniqueness argument to nonsimple (but nuclear) domains appears in [45, Lemma 5.9].

Theorem 3.5. *Let C be a separable, unital, exact C^*-algebra that satisfies the UCT. Let $\Delta : C^1_+\backslash\{0\} \to \mathbb{N}$ be a control function, let $\mathcal{G} \subset C$ be a finite subset, and let $\delta > 0$. Then there exists $n \in \mathbb{N}$ such that, for any admissible target algebra D of finite type, any unital Δ-full *-homomorphism $\iota : C \to D$, and any nuclear *-homomorphisms $\phi, \psi : C \to D$, if*

(i) $\phi_* = \psi_* : \underline{K}(C) \to \underline{K}(D)$, and
(ii) $\phi(1_C)$ is unitarily equivalent to $\psi(1_C)$,

then there exists a unitary $u \in M_{n+1}(D)$ such that

$$(3.6) \qquad \|u(\phi(c) \oplus \iota^{\oplus n}(c))u^* - (\psi(c) \oplus \iota^{\oplus n}(c))\| < \delta, \quad c \in \mathcal{G}.$$

Proof. Suppose that the statement is false, so it fails for some control function Δ, finite set $\mathcal{G}$ and $\delta > 0$. Then for every $n \in \mathbb{N}$, there exists an admissible target algebra of finite type D_n and *-homomorphisms ι_n, ϕ_n, ψ_n such that ι_n is unital and Δ-full, ϕ_n, ψ_n are nuclear, $\phi_n(1_C)$ is unitarily equivalent to $\psi_n(1_C)$, $(\phi_n)_* = (\psi_n)_*$, yet

$$(3.7) \qquad \max_{c \in \mathcal{G}} \|v(\phi_n(c) \oplus \iota_n^{\oplus n}(c))v^* - (\psi_n(c) \oplus \iota_n^{\oplus n}(c))\| \geq \delta$$

for every unitary $v \in M_{n+1}(D_n)$.

Define

$$(3.8) \qquad D_\infty := \prod_{n=1}^{\infty} D_n/\{(d_n)_{n=1}^{\infty} \mid \lim_{n\to\infty} \|d_n\| = 0\},$$

and define $\Phi, \Psi, I : C \to D_\infty$ to be the *-homomorphisms induced by the sequences $(\phi_n), (\psi_n), (\iota_n)$ respectively. Since each ϕ_n and ψ_n is nuclear, and C is exact, Φ, Ψ are nuclear by [19, Proposition 3.3]. As C satisfies the UCT, $[\Phi] = [\Psi]$ in $KK(C, D_\infty)$ by Proposition 3.3. Each $\phi_n(1_C)$ is unitarily equivalent to $\psi_n(1_C)$, so that $\Phi(1_C)$ is unitarily equivalent to $\Psi(1_C)$.

Since each ι_n is Δ-full, it follows that I is also Δ-full. Indeed, given a nonzero positive contraction $c \in C$, there are contractions $d_{1,n}, \ldots, d_{\Delta(c),n} \in D_n$ such that $1_{D_n} = \sum_{i=1}^{\Delta(c)} d_{i,n}^* \iota_n(c)d_{i,n}$. Thus letting d_i be the contraction in D_∞ represented by $(d_{i,n})_n$, we have $1_{D_\infty} = \sum_{i=1}^{\Delta(c)} d_i^* I(c)d_i$. Consequently $I(c)$ is full in D_∞ for every nonzero positive contraction $c \in C$. The same holds for all nonzero $c \in C$, as $I(|c|)$ is in the ideal generated by $I(c)$. That is, I is totally full.

Hence by Theorem 3.1, there exist $m \in \mathbb{N}$ and a unitary $u \in M_{m+1}(D_\infty)$ such that

$$(3.9) \qquad \|u(\Phi(c) \oplus I^{\oplus m}(c))u^* - (\Psi(c) \oplus I^{\oplus m}(c))\| < \delta, \quad c \in \mathcal{G}.$$

The unitary u lifts to a sequence of unitaries $u_n \in M_{m+1}(D_n)$, and for n sufficiently large, it follows that

$$(3.10) \qquad \|u_n(\phi_n(c) \oplus \iota_n^{\oplus m}(c))u_n^* - (\psi_n(c) \oplus \iota_n^{\oplus m}(c))\| < \delta, \quad c \in \mathcal{G}.$$

Taking $n \geq m$ large enough so that (3.10) holds, the unitary $v := u_n \oplus 1_{D_n}^{\oplus(n-m)}$ in $M_{n+1}(D_n)$ gives a contradiction to (3.7), proving the theorem. □

4. A PATCHING LEMMA

In this section we provide the patching lemma, which connects two *-homomorphisms ν_0 and ν_1 defined on the suspension $C_0((0,1), A)$ of A into a single map ρ, in the sense that ρ recovers the two original maps on the left and right thirds of the interval respectively. The map ρ will only asymptotically be a *-homomorphism, with the estimates depending on how close the original maps are to being approximately unitarily equivalent over the middle third of the interval.

Lemma 4.1. *There exists a partition of unity $f_0, f_{\frac{1}{2}}, f_1$ for $C([0,1])$ with $f_{\frac{1}{2}} \in C_0((\frac{1}{3}, \frac{2}{3}))$, such that the following holds.*

Let A and E be unital C*-algebras and $\theta : C([0,1]) \to E$ a unital *-homomorphism. Given *-homomorphisms $\nu_0, \nu_1 : C_0((0,1), A) \to E$ compatible with θ and a unitary $v \in E$, there exists a completely positive map $\rho : C_0((0,1), A) \to M_2(E) = E \otimes M_2$ such that:

(i) $\rho|_{C_0((0,\frac{1}{3}),A)} = \begin{pmatrix} \nu_0|_{C_0((0,\frac{1}{3}),A)} & 0 \\ 0 & 0 \end{pmatrix}$.

(ii) $\rho|_{C_0((\frac{2}{3},1),A)} = \begin{pmatrix} \nu_1|_{C_0((\frac{2}{3},1),A)} & 0 \\ 0 & 0 \end{pmatrix}$.

(iii) If $\tau \in T(E)$ satisfies $\tau \circ \nu_0 = \tau \circ \nu_1$, then $(\tau \otimes \mathrm{Tr}_2) \circ \rho = \tau \circ \nu_0$, where we recall that Tr_2 is the canonical non-normalised tracial functional on M_2.

(iv) For $x \in C_0((0,1), A)$ and $h \in C([0,1])$,

$$(4.1) \qquad \|\rho(hx) - \theta^{\oplus 2}(h)\rho(x)\| \leq (\|[\theta(f_{\frac{1}{2}}), v]\|\|h\| + \|[\theta(hf_{\frac{1}{2}}), v]\|)\|x\|.$$

(v) For $x, y \in C_0((0,1), A)$,

$$\|\rho(xy) - \rho(x)\rho(y)\|$$
$$\leq \left(7\|[v, \theta(f_{\frac{1}{2}})]\| + 2\|[v, \theta(f_0 f_{\frac{1}{2}})]\| + 5\|[v, \theta(f_{\frac{1}{2}} f_1)]\| \right) \|x\|\|y\|$$
$$+ \left(\|\nu_1(f_{\frac{1}{2}} f_1 x)v - v\nu_0(f_{\frac{1}{2}} f_1 x)\| \right.$$
$$(4.2) \qquad \left. + \|\nu_1(f_{\frac{1}{2}} x)v - v\nu_0(f_{\frac{1}{2}} x)\| \right) \|y\|.$$

Remark 4.2. Note that in (iv) and (v), the estimates on the right-hand sides are bounded in terms of how well $\mathrm{Ad}(v) \circ \nu_0|_{C_0((\frac{1}{3},\frac{2}{3}),A)}$ approximates $\nu_1|_{C_0((\frac{1}{3},\frac{2}{3}),A)}$. Thus, for example, if

$$(4.3) \qquad \mathrm{Ad}(v) \circ \nu_0|_{C_0((\frac{1}{3},\frac{2}{3}),A)} = \nu_1|_{C_0((\frac{1}{3},\frac{2}{3}),A)}$$

then (iv) and (v) tell us that ρ is a *-homomorphism that is compatible with θ giving an exact patching result.

Likewise, given finite sets $\mathcal{F} \subset C_0((0,1), A)$, $\mathcal{F}' \subset C([0,1])$, and a tolerance $\eta > 0$, there are a finite set $\mathcal{G} \subset C_0((\frac{1}{3}, \frac{2}{3}), A)$ and $\delta > 0$ such that,

if

$$\text{(4.4)} \qquad \text{Ad}(v) \circ \nu_0(x) \approx_\delta \nu_1(x), \quad x \in \mathcal{G},$$

then

$$\rho(xy) \approx_\eta \rho(x)\rho(y), \quad x, y \in \mathcal{F}, \text{ and}$$
$$\text{(4.5)} \qquad \rho(hx) \approx_\eta \theta^{\oplus 2}(h)\rho(x) \quad x \in \mathcal{F}, \ h \in \mathcal{F}'.$$

Namely, set

$$\mathcal{G} := \{f_{\frac{1}{2}}, f_0 f_{\frac{1}{2}}, f_1 f_{\frac{1}{2}}\} \cup \{hf_{\frac{1}{2}} \mid h \in \mathcal{F}'\}$$
$$\text{(4.6)} \qquad \cup \{f_{\frac{1}{2}} x \mid x \in \mathcal{F}\} \cup \{f_{\frac{1}{2}} f_1 x \mid x \in \mathcal{F}\}$$

and take δ such that

$$\text{(4.7)} \qquad (M' + 1)M\delta \leq \eta \text{ and } 14M^2\delta + 2M\delta \leq \eta,$$

where $M := \max\{\|x\| \mid x \in \mathcal{F}\}$ and $M' := \max\{\|h\| \mid h \in \mathcal{F}'\}$.

Proof of Lemma 4.1. Define functions $f_0, f_1 \in C([0, 1])$ so that

$$\text{(4.8)} \qquad f_0|_{[0, \frac{1}{3}]} \equiv 1, \qquad\qquad f_0|_{[\frac{2}{5}, 1]} \equiv 0,$$
$$\text{(4.9)} \qquad f_1|_{[0, \frac{3}{5}]} \equiv 0, \qquad\qquad f_1|_{[\frac{2}{3}, 1]} \equiv 1,$$

and f_0, f_1 are linear on the complements of the intervals in (4.8) and (4.9) respectively. Write

$$\text{(4.10)} \qquad f_{\frac{1}{2}} := 1_{C([0,1])} - f_0 - f_1$$

so that $f_0, f_{\frac{1}{2}}, f_1$ comprise a partition of unity of $C([0, 1])$ and $f_{\frac{1}{2}} \in C_0((\frac{1}{3}, \frac{2}{3}))$. Let $U \in C([0, 1], M_2) \cong C([0, 1]) \otimes M_2$ be a unitary which satisfies

$$\text{(4.11)} \qquad U|_{[0, \frac{2}{5}]} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U|_{[\frac{3}{5}, 1]} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
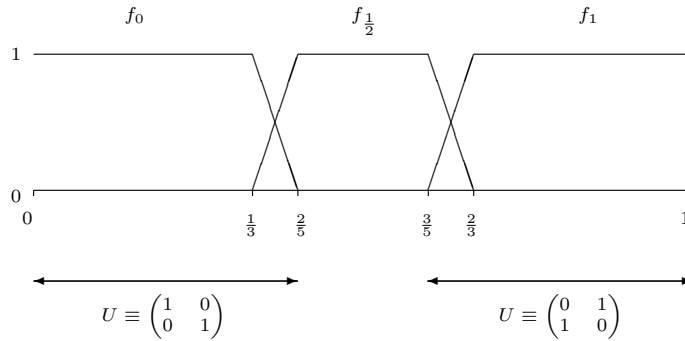
These objects are summarised in Figure 2.

FIGURE 2. f_0, $f_{\frac{1}{2}}$, f_1 and the unitary U

Given E, a unital *-homomorphism $\theta : C([0, 1]) \to E$, compatible *-homomorphisms $\nu_i : C_0((0, 1), A) \to E$ and a unitary $v \in E$, define c.p.c.

maps $\rho_0, \rho_{\frac{1}{2}}, \rho_1 : C_0((0,1), A) \to M_2(E)$ by

$$(4.12) \qquad \rho_i(x) := \begin{pmatrix} \nu_i(f_i x) & 0 \\ 0 & 0 \end{pmatrix}, \quad x \in C_0((0,1), A), \ i = 0, 1 \text{ and}$$

$$(4.13) \qquad \rho_{\frac{1}{2}}(x) := V \begin{pmatrix} \nu_0(f_{\frac{1}{2}} x) & 0 \\ 0 & 0 \end{pmatrix} V^*, \quad x \in C_0((0,1), A),$$

where

$$(4.14) \qquad V := \left((\theta \otimes \mathrm{id}_{M_2})(U^*) \begin{pmatrix} 1_E & 0 \\ 0 & v \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U) \right),$$

a unitary in $M_2(E)$. Then define

$$(4.15) \qquad \rho := \rho_0 + \rho_{\frac{1}{2}} + \rho_1.$$

This is certainly a completely positive map, and the location of the supports of $f_0, f_{\frac{1}{2}}, f_1$ ensure that (i) and (ii) hold. For (iii), suppose that $\tau \in T(E)$ has $\tau \circ \nu_0 = \tau \circ \nu_1$. Then

$$(\tau \otimes \mathrm{Tr}_2)(\rho(x)) \quad = \quad (\tau \otimes \mathrm{Tr}_2)(\rho_0(x) + \rho_{\frac{1}{2}}(x) + \rho_1(x))$$

$$\overset{(4.12),(4.13)}{=} \tau(\nu_0(f_0 x) + \nu_0(f_{\frac{1}{2}} x) + \nu_1(f_1 x))$$

$$\overset{\tau \circ \nu_0 = \tau \circ \nu_1}{=} \tau \circ \nu_0(f_0 x + f_{\frac{1}{2}} x + f_1 x)$$

$$(4.16) \qquad \overset{(4.10)}{=} \tau \circ \nu_0(x), \quad x \in C_0((0,1), A),$$

establishing (iii).

For (iv), fix $h \in C([0,1])$ and $x \in C_0((0,1), A)$; by rescaling, we may assume these to be contractions. Then compatibility gives

$$(4.17) \qquad \rho_{\frac{1}{2}}(hx) = V \begin{pmatrix} \theta(f_{\frac{1}{2}} h) & 0 \\ 0 & \theta(f_{\frac{1}{2}} h) \end{pmatrix} \begin{pmatrix} \nu_0(x) & 0 \\ 0 & 0 \end{pmatrix} V^*,$$

whence

$$\| \rho_{\frac{1}{2}}(hx) - \theta^{\oplus 2}(h) \rho_{\frac{1}{2}}(x) \|$$

$$\leq \| [V, \theta^{\oplus 2}(f_{\frac{1}{2}} h)] \| + \| [V, \theta^{\oplus 2}(f_{\frac{1}{2}})] \|$$

$$(4.18) \qquad = \| [v, \theta(h f_{\frac{1}{2}})] \| + \| [v, \theta(f_{\frac{1}{2}})] \|,$$

as

$$(4.19) \qquad [\theta^{\oplus 2}(C([0,1])), (\theta \otimes \mathrm{id}_2)(U)] = 0.$$

For $i = 0, 1$, the compatibility of ν_i with θ gives

$$(4.20) \qquad \rho_i(hx) = \theta^{\oplus 2}(h) \rho_i(x).$$

Combining this with (4.18) gives

$$\| \rho(hx) - \theta^{\oplus 2}(h) \rho(x) \|$$

$$\leq \sum_{i=0,\frac{1}{2},1} \| \rho_i(hx) - \theta^{\oplus 2}(h) \rho_i(x) \|$$

$$(4.21) \qquad \leq \| [v, \theta(h f_{\frac{1}{2}})] \| + \| [v, \theta(f_{\frac{1}{2}})] \|,$$

establishing (iv).

Finally we show (v); by rescaling, it again suffices to do this when $x, y \in C_0((0,1), A)$ are contractions. The proof amounts to a long, but routine, calculation. We shall estimate

$$(4.22) \qquad \|\rho_i(x)\rho_j(y) - \theta^{\oplus 2}(f_i)\rho_j(xy)\|, \quad i, j = 0, \tfrac{1}{2}, 1.$$

There are nine cases to consider. For $i = j = 0$ or $i = j = 1$, compatibility of ν_i with θ gives

$$(4.23) \qquad \theta^{\oplus 2}(f_i)\rho_i(xy) = \rho_i(x)\rho_i(y),$$

while, for $i, j \in \{0, 1\}$ with $i \neq j$, compatibility combines with $f_0 f_1 = 0$, to yield

$$(4.24) \qquad \theta^{\oplus 2}(f_i)\rho_j(xy) = 0 = \rho_i(x)\rho_j(y).$$

Thus (4.22) evaluates to zero when $i, j \in \{0, 1\}$.

When $i = j = \tfrac{1}{2}$, compatibility gives

$$(4.25) \qquad \rho_{\frac{1}{2}}(x)\rho_{\frac{1}{2}}(y) = V \begin{pmatrix} \theta(f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_{\frac{1}{2}}) \end{pmatrix} \begin{pmatrix} \nu_0(f_{\frac{1}{2}}xy) & 0 \\ 0 & 0 \end{pmatrix} V^*,$$

while

$$(4.26) \qquad \theta^{\oplus 2}(f_{\frac{1}{2}})\rho_{\frac{1}{2}}(xy) = \begin{pmatrix} \theta(f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_{\frac{1}{2}}) \end{pmatrix} V \begin{pmatrix} \nu_0(f_{\frac{1}{2}}xy) & 0 \\ 0 & 0 \end{pmatrix} V^*.$$

Using these and (4.19), we have

$$\|\rho_{\frac{1}{2}}(x)\rho_{\frac{1}{2}}(y) - \theta^{\oplus 2}(f_{\frac{1}{2}})\rho_{\frac{1}{2}}(xy)\| \leq \|[\theta^{\oplus 2}(f_{\frac{1}{2}}), V]\|$$
$$(4.27) \qquad\qquad\qquad = \|[\theta(f_{\frac{1}{2}}), v]\|.$$

For the cases in which $\{i, j\} = \{0, \tfrac{1}{2}\}$, first note that (4.8) and (4.11) give

$$\begin{pmatrix} \theta(f_0) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \theta(f_0) & 0 \\ 0 & 0 \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U)$$
$$(4.28) \qquad\qquad = (\theta \otimes \mathrm{id}_{M_2})(U) \begin{pmatrix} \theta(f_0) & 0 \\ 0 & 0 \end{pmatrix},$$

and therefore, using (4.14),

$$(4.29) \qquad \begin{pmatrix} \theta(f_0) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \theta(f_0) & 0 \\ 0 & 0 \end{pmatrix} V = V \begin{pmatrix} \theta(f_0) & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\rho_{\frac{1}{2}}(x)\rho_0(y) = V \begin{pmatrix} \theta(f_{\frac{1}{2}})\nu_0(x) & 0 \\ 0 & 0 \end{pmatrix} V^* \begin{pmatrix} \theta(f_0)\nu_0(y) & 0 \\ 0 & 0 \end{pmatrix}$$

$$\overset{(4.29)}{=} V \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}})\nu_0(xy) & 0 \\ 0 & 0 \end{pmatrix}$$

$$\overset{(4.29)}{=} \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}})\nu_0(xy) & 0 \\ 0 & 0 \end{pmatrix}$$

$$(4.30) \qquad = \theta^{\oplus 2}(f_{\frac{1}{2}})\rho_0(xy),$$

and this shows that (4.22) vanishes for $(i,j) = (\frac{1}{2}, 0)$. Next compute

$$\begin{pmatrix} \theta(f_0) & 0 \\ 0 & \theta(f_0) \end{pmatrix} V \begin{pmatrix} \theta(f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_{\frac{1}{2}}) \end{pmatrix}$$

$$\overset{(4.8),(4.11),(4.14)}{=} \begin{pmatrix} \theta(f_0) & 0 \\ 0 & \theta(f_0)v \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U) \begin{pmatrix} \theta(f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_{\frac{1}{2}}) \end{pmatrix}$$

$$\overset{(4.19)}{=} \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_0)v\theta(f_{\frac{1}{2}}) \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U)$$

$$\approx_{\|[v,\theta(f_{\frac{1}{2}})]\|} \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_0 f_{\frac{1}{2}})v \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U)$$

$$\approx_{\|[v,\theta(f_0 f_{\frac{1}{2}})]\|} \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}}) & 0 \\ 0 & v\theta(f_0 f_{\frac{1}{2}}) \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U)$$

$$\overset{(4.8),(4.11)}{=} \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}}) & 0 \\ 0 & v\theta(f_0 f_{\frac{1}{2}}) \end{pmatrix}$$

$$\approx_{\|[v,\theta(f_0 f_{\frac{1}{2}})]\|} \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_0 f_{\frac{1}{2}})v \end{pmatrix}$$

$$(4.31) \quad \approx_{\|[v,\theta(f_{\frac{1}{2}})]\|} \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_0)v\theta(f_{\frac{1}{2}}) \end{pmatrix}.$$

Then with $\varepsilon_0 := 2(\|[v, \theta(f_{\frac{1}{2}})]\| + \|[v, \theta(f_0 f_{\frac{1}{2}})]\|)$, we have

$$\rho_0(x)\rho_{\frac{1}{2}}(y) = \begin{pmatrix} \nu_0(x)\theta(f_0) & 0 \\ 0 & 0 \end{pmatrix} V \begin{pmatrix} \theta(f_{\frac{1}{2}})\nu_0(y) & 0 \\ 0 & 0 \end{pmatrix} V^*$$

$$\overset{(4.29)}{=} \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}})\nu_0(xy) & 0 \\ 0 & 0 \end{pmatrix} V^*$$

$$= \begin{pmatrix} \theta(f_0 f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_0)v\theta(f_{\frac{1}{2}}) \end{pmatrix} \begin{pmatrix} \nu_0(xy) & 0 \\ 0 & 0 \end{pmatrix} V^*$$

$$\overset{(4.31)}{\approx_{\varepsilon_0}} \begin{pmatrix} \theta(f_0) & 0 \\ 0 & \theta(f_0) \end{pmatrix} V \begin{pmatrix} \nu_0(f_{\frac{1}{2}}xy) & 0 \\ 0 & 0 \end{pmatrix} V^*$$

$$(4.32) \qquad = \theta^{\oplus 2}(f_0)\rho_{\frac{1}{2}}(xy).$$

For the cases when $\{i, j\} = \{\frac{1}{2}, 1\}$, similarly to (4.31), we compute

$$\begin{pmatrix} \theta(f_1) & 0 \\ 0 & \theta(f_1) \end{pmatrix} V \begin{pmatrix} \theta(f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_{\frac{1}{2}}) \end{pmatrix}$$

$$\overset{(4.9),(4.11),(4.14)}{=} \begin{pmatrix} 0 & \theta(f_1)v \\ \theta(f_1) & 0 \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U) \begin{pmatrix} \theta(f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_{\frac{1}{2}}) \end{pmatrix}$$

$$\overset{(4.19)}{=} \begin{pmatrix} 0 & \theta(f_1)v\theta(f_{\frac{1}{2}}) \\ \theta(f_1 f_{\frac{1}{2}}) & 0 \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U)$$

$$\approx_{\|[v,\theta(f_{\frac{1}{2}})]\|} \begin{pmatrix} 0 & \theta(f_1 f_{\frac{1}{2}})v \\ \theta(f_1 f_{\frac{1}{2}}) & 0 \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U)$$

$$\approx_{\|[v,\theta(f_1 f_{\frac{1}{2}})]\|} \begin{pmatrix} 0 & v\theta(f_1 f_{\frac{1}{2}}) \\ \theta(f_1 f_{\frac{1}{2}}) & 0 \end{pmatrix} (\theta \otimes \mathrm{id}_{M_2})(U)$$

$$(4.33) \quad \overset{(4.9),(4.11)}{=} \quad \begin{pmatrix} v\theta(f_1 f_{\frac{1}{2}}) & 0 \\ 0 & \theta(f_1 f_{\frac{1}{2}}) \end{pmatrix}$$

$$(4.34) \quad \approx_{\|[v,\theta(f_1 f_{\frac{1}{2}})]\|} \begin{pmatrix} \theta(f_1 f_{\frac{1}{2}})v & 0 \\ 0 & \theta(f_1 f_{\frac{1}{2}}) \end{pmatrix}.$$

With $\varepsilon_1 := \|[v,\theta(f_{\frac{1}{2}})]\| + 2\|[v,\theta(f_1 f_{\frac{1}{2}})]\|$, this gives

$$\rho_1(x)\rho_{\frac{1}{2}}(y) \quad = \quad \begin{pmatrix} \nu_1(f_1 x) & 0 \\ 0 & 0 \end{pmatrix} V \begin{pmatrix} \nu_0(f_{\frac{1}{2}}y) & 0 \\ 0 & 0 \end{pmatrix} V^*$$

$$\overset{(4.34)}{\approx_{\varepsilon_1}} \quad \begin{pmatrix} \theta(f_1)\nu_1(f_{\frac{1}{2}}x)v\nu_0(y) & 0 \\ 0 & 0 \end{pmatrix} V^*$$

$$(4.35) \quad \overset{(4.9),(4.11),(4.14)}{=} (\theta \otimes \mathrm{id}_{M_2})(U^*) \begin{pmatrix} 0 & 0 \\ \nu_1(f_1 f_{\frac{1}{2}}x)v\nu_0(y) & 0 \end{pmatrix} V^*,$$

whereas

$$\rho_{\frac{1}{2}}(f_1 xy) \quad = \quad V \begin{pmatrix} \nu_0(f_1 f_{\frac{1}{2}}xyb) & 0 \\ 0 & 0 \end{pmatrix} V^*$$

$$(4.36) \quad \overset{(4.9),(4.11),(4.14)}{=} (\theta \otimes \mathrm{id}_{M_2})(U^*) \begin{pmatrix} 0 & 0 \\ v\nu_0(f_1 f_{\frac{1}{2}}x)\nu_0(y) & 0 \end{pmatrix} V^*.$$

Putting these together and using (4.18) with $h := f_1$, we obtain

$$\|\rho_1(x)\rho_{\frac{1}{2}}(y) - \theta^{\oplus 2}(f_1)\rho_{\frac{1}{2}}(xy)\|$$
$$\leq \|\rho_1(x)\rho_{\frac{1}{2}}(y) - \rho_{\frac{1}{2}}(f_1 xy)\|$$
$$\quad + \|\rho_{\frac{1}{2}}(f_1 xy) - \theta^{\oplus 2}(f_1)\rho_{\frac{1}{2}}(xy)\|$$
$$\leq \varepsilon_1 + \|\nu_1(f_{\frac{1}{2}}f_1 x)v - v\nu_0(f_{\frac{1}{2}}f_1 x)\|$$
$$\quad + \|[v,\theta(f_1 f_{\frac{1}{2}})]\| + \|[v,\theta(f_{\frac{1}{2}})]\|$$
$$= \|\nu_1(f_{\frac{1}{2}}f_1 x)v - v\nu_0(f_{\frac{1}{2}}f_1 x)\|$$
$$(4.37) \quad\quad + 3\|[v,\theta(f_1 f_{\frac{1}{2}})]\| + 2\|[v,\theta(f_{\frac{1}{2}})]\|.$$

For the final case, we compute

$$V \begin{pmatrix} \theta(f_{\frac{1}{2}}f_1) & 0 \\ 0 & 0 \end{pmatrix} \quad \overset{(4.9),(4.11),(4.14)}{=} \quad (\theta \otimes \mathrm{id}_{M_2})(U^*) \begin{pmatrix} 0 & 0 \\ v\theta(f_{\frac{1}{2}}f_1) & 0 \end{pmatrix}$$

$$\approx_{\|[\theta(f_{\frac{1}{2}}f_1),v]\|} (\theta \otimes \mathrm{id}_{M_2})(U^*) \begin{pmatrix} 0 & 0 \\ \theta(f_{\frac{1}{2}}f_1)v & 0 \end{pmatrix}$$

$$(4.38) \quad \overset{(4.9),(4.11),(4.14)}{=} \quad \begin{pmatrix} \theta(f_{\frac{1}{2}}f_1)v & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, with $\varepsilon_1' := \|[v, \theta(f_{\frac{1}{2}})]\| + \|[v, \theta(f_1 f_{\frac{1}{2}})]\|$, we have

$$
\rho_{\frac{1}{2}}(x)\rho_1(y) \;=\; V \begin{pmatrix} \nu_0(f_{\frac{1}{2}}x) & 0 \\ 0 & 0 \end{pmatrix} V^* \begin{pmatrix} \nu_1(f_1 y) & 0 \\ 0 & 0 \end{pmatrix}
$$

$$
\overset{(4.33)}{\approx_{\varepsilon_1'}} \quad V \begin{pmatrix} \nu_0(x) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta(f_{\frac{1}{2}} f_1) v^* \nu_1(y) & 0 \\ 0 & \theta(f_{\frac{1}{2}} f_1) \end{pmatrix}
$$

$$
= \quad V \begin{pmatrix} \theta(f_{\frac{1}{2}} f_1) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu_0(x) v^* \nu_1(y) & 0 \\ 0 & 0 \end{pmatrix}
$$

$$
(4.39) \qquad \overset{(4.38)}{\approx_{\|[\theta(f_{\frac{1}{2}} f_1), v]\|}} \begin{pmatrix} \theta(f_1)\theta(f_{\frac{1}{2}}) v \nu_0(x) v^* \nu_1(y) & 0 \\ 0 & 0 \end{pmatrix},
$$

while

$$
(4.40) \qquad \theta^{\oplus 2}(f_{\frac{1}{2}})\rho_1(xy) = \begin{pmatrix} \theta(f_1)\nu_1(f_{\frac{1}{2}}x)\nu_1(y) & 0 \\ 0 & 0 \end{pmatrix}.
$$

Putting these together yields the estimate

$$
\|\rho_{\frac{1}{2}}(x)\rho_1(y) - \theta^{\oplus 2}(f_{\frac{1}{2}}))\rho_1(xy)\|
$$
$$
\leq \varepsilon_1' + \|[\theta(f_1 f_{\frac{1}{2}}), v]\|
$$
$$
\qquad + \|\theta(f_{\frac{1}{2}}) v \nu_0(x) v^* - \nu_1(f_{\frac{1}{2}}x)\|
$$
$$
\leq \varepsilon_1' + \|[\theta(f_1 f_{\frac{1}{2}}), v]\|
$$
$$
\qquad + \|[\theta(f_{\frac{1}{2}}), v]\| + \|v\nu_0(f_{\frac{1}{2}}x) - \nu_1(f_{\frac{1}{2}}x)v\|
$$
$$
= 2\|[\theta(f_{\frac{1}{2}}), v]\| + 2\|[\theta(f_1 f_{\frac{1}{2}}), v]\|
$$
$$
(4.41) \qquad + \|v\nu_0(f_{\frac{1}{2}}x) - \nu_1(f_{\frac{1}{2}}x)v\|.
$$

In conclusion, (4.22) vanishes for $(i,j) = (0,1), (1,0), (0,0), (1,1), (\frac{1}{2}, 0)$ and in the other four cases $(i,j) = (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}), (1, \frac{1}{2}), (\frac{1}{2}, 1)$, we have the estimates (4.27), (4.32), (4.37) and (4.41). Thus

$$
\|\rho(x)\rho(y) - \rho(xy)\|
$$
$$
\overset{(4.10),(4.15)}{=} \left\| \sum_{i,j=0,\frac{1}{2},1} \rho_i(x)\rho_j(y) - \sum_{i,j=0,\frac{1}{2},1} \theta^{\oplus 2}(f_i)\rho_j(ab) \right\|
$$
$$
\leq \quad \sum_{i,j=0,\frac{1}{2},1} \|\rho_i(x)\rho_j(y) - \theta^{\oplus 2}(f_i)\rho_j(xy)\|
$$
$$
\leq \quad 7\|[v, \theta(f_{\frac{1}{2}})]\| + 2\|[v, \theta(f_0 f_{\frac{1}{2}})]\| + 5\|[v, \theta(f_{\frac{1}{2}} f_1)]\|
$$
$$
\qquad + \|\nu_1(f_{\frac{1}{2}} f_1 x)v - v\nu_0(f_{\frac{1}{2}} f_1 x)\|
$$
$$
(4.42) \qquad + \|\nu_1(f_{\frac{1}{2}}x)v - v\nu_0(f_{\frac{1}{2}}x)\|,
$$

establishing (v) (recalling that x and y are assumed to be contractions). $\quad\square$

5. Proof of Theorem A

In this section we give the proof of our main result. By Remark 1.5 (iii) (and since a C*-algebra satisfies the UCT if and only if its smallest unitisation does), it suffices to prove the theorem when A is unital. So let A be a separable, unital and nuclear C*-algebra in the UCT class and let $\tau_A \in T(A)$ be faithful. Fix a finite subset $\mathcal{F}_A \subset A$ of contractions containing 1_A and a tolerance $\varepsilon > 0$. We will produce $N \in \mathbb{N}$ and a completely positive map $\Psi : A \to \mathcal{Q}_\omega \otimes M_{2N} \cong \mathcal{Q}_\omega$ such that

$$(5.1) \qquad \|\Psi(ab) - \Psi(a)\Psi(b)\| < \varepsilon, \quad a, b \in \mathcal{F}_A,$$

$$(5.2) \qquad \tfrac{1}{2}\tau_A(a) = (\tau_{\mathcal{Q}_\omega} \otimes \tau_{M_{2N}})(\Psi(a)), \quad a \in A.$$

Once this is achieved, quasidiagonality of τ_A follows from the characterisation in Proposition 1.4 (iic).

Define functions $f, \grave{g}, \acute{g}, \grave{g}_{-1}, \acute{g}_{+1} \in C([0,1])$ by

$$f|_{[0,\frac{1}{9}]} \equiv 0, \qquad\qquad f|_{[\frac{2}{9},\frac{7}{9}]} \equiv 1, \qquad f|_{[\frac{8}{9},1]} \equiv 0,$$

$$\grave{g}|_{[0,\frac{8}{9}]} \equiv 1, \qquad\qquad \grave{g}(1) = 0,$$

$$\acute{g}(0) = 0, \qquad\qquad \acute{g}|_{[\frac{1}{9},1]} \equiv 1,$$

$$\grave{g}_{-1}|_{[0,\frac{2}{9}]} \equiv 1, \qquad\qquad \grave{g}_{-1}|_{[\frac{1}{3},1]} \equiv 0,$$

$$(5.3) \qquad \acute{g}_{+1}|_{[0,\frac{2}{3}]} \equiv 0, \qquad\qquad \acute{g}_{+1}|_{[\frac{7}{9},1]} \equiv 1,$$

and such that each function is linear on the complements of the intervals used to define it. These functions are shown in Figure 3.
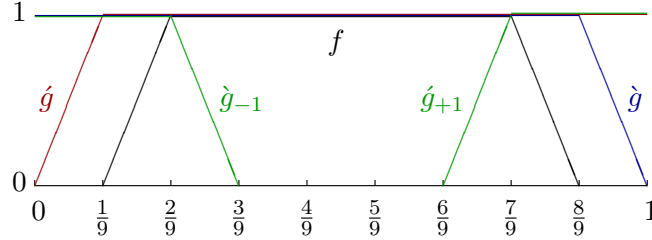

FIGURE 3. f, $\grave{g}$, $\acute{g}$, $\grave{g}_{-1}$, $\acute{g}_{+1}$

Define

$$\mathcal{F} := \{f \otimes a \mid a \in \mathcal{F}_A\} \cup \{f \otimes ab \mid a, b \in \mathcal{F}_A\} \subset C_0((0,1), A),$$

$$(5.4) \qquad \mathcal{F}' := \{f, \grave{g}, \acute{g}, \grave{g}_{-1}, \acute{g}_{+1}\} \subset C([0,1])$$

and choose

$$(5.5) \qquad\qquad\qquad 0 < \eta < \tfrac{\varepsilon}{24}.$$

As in Remark 4.2, use Lemma 4.1 to find a finite subset $\mathcal{G} \subset C_0((\frac{1}{3}, \frac{2}{3}), A)$ and $\delta > 0$ such that Property 5.1 below is satisfied. Note that we change from the matrix notation of Lemma 4.1 (i), (ii) to $\oplus$-notation in (i), (ii) below; see Notation 1.8.

Property 5.1. *Let E be a unital C*-algebra with trace τ_E. Let $\theta : C([0,1]) \to E$ be a unital *-homomorphism and let $\nu_0, \nu_1 : C_0((0,1), A) \to E$ be compatible *-homomorphisms satisfying $\tau_E \circ \nu_i = \tau_{\mathrm{Leb}} \otimes \tau_A$ for $i = 0, 1$, where τ_{Leb}*

is the Lebesgue trace defined in (2.23). Suppose there exists a unitary $v \in E$ satisfying

$$(5.6) \qquad v\nu_0(x)v^* \approx_\delta \nu_1(x), \quad x \in \mathcal{G}.$$

Then there exists a completely positive map $\rho : C_0((0,1), A) \to M_2(E) \cong E \otimes M_2$ such that

(i) $\rho|_{C_0((0,\frac{1}{3}),A)} = \nu_0|_{C_0((0,\frac{1}{3}),A)} \oplus 0_E$,
(ii) $\rho|_{C_0((\frac{2}{3},1),A)} = \nu_1|_{C_0((\frac{2}{3},1),A)} \oplus 0_E$,
(iii) $(\tau_E \otimes \mathrm{Tr}_2) \circ \rho = \tau_{\mathrm{Leb}} \otimes \tau_A$,
(iv) $\rho(ab) \approx_\eta \rho(a)\rho(b)$, *for $a, b \in \mathcal{F}$, and*
(v) $\rho(ha) \approx_\eta \theta^{\oplus 2}(h)\rho(a)$, *for $a \in \mathcal{F}$, $h \in \mathcal{F}'$.*

Define

$$(5.7) \qquad C := \{x \in C([0,1], A) \mid \exists \lambda \in \mathbb{C} \text{ s.t. } x|_{[0,\frac{1}{3}]} \equiv x|_{[\frac{2}{3},1]} \equiv \lambda 1_A\}.$$

As an abstract algebra, C is isomorphic to $C_0((\frac{1}{3}, \frac{2}{3}), A)^{\sim}$, and so, as noted after Definition 1.7, C satisfies the UCT. However we regard C as the sub-algebra of $C([0,1], A)$ given in (5.7) so that $\tau_{\mathrm{Leb}} \otimes \tau_A$ restricts to a trace on C.

Since τ_A and τ_{Leb} are faithful traces, so too is $\tau_{\mathrm{Leb}} \otimes \tau_A$ by Kirchberg's slice lemma (see [58, Lemma 4.1.9] in Rørdam's book for a published version). Thus we may define a control function $\Delta : C_+^1 \backslash \{0\} \to \mathbb{N}$ by taking $\Delta(c)$ to be a square number such that

$$(5.8) \qquad (\tau_{\mathrm{Leb}} \otimes \tau_A)(c) > \frac{2}{\sqrt{\Delta(c)}}.$$

Apply the stable uniqueness theorem (Theorem 3.5) to C with respect to the control function Δ, and the finite set $\mathcal{G}$ and tolerance δ of Property 5.1, to obtain $n \in \mathbb{N}$ satisfying Property 5.2 below. Note that since C is nuclear, so are all *-homomorphisms whose domain is C. Also recall that $\mathcal{Q}_\omega$ is an admissible target algebra of finite type.

Property 5.2. *Let D be a C*-algebra isomorphic to $\mathcal{Q}_\omega$, $\iota : C \to D$ a Δ-full unital *-homomorphism and $\phi, \psi : C \to D$ unital *-homomorphisms satisfying $\phi_* = \psi_* : \underline{K}(C) \to \underline{K}(D)$. Then there exists a unitary $u \in M_{n+1}(D)$ such that*

$$(5.9) \qquad u(\phi(x) \oplus \iota^{\oplus n}(x))u^* \approx_\delta \psi(x) \oplus \iota^{\oplus n}(x), \quad x \in \mathcal{G}.$$

In terms of this fixed integer n, define integers

$$(5.10) \qquad N := 2n, \quad m := 4n + 1,$$

and relatively open intervals $I_i \subset [0,1]$ for $i = 0, \ldots, N+1$ by

$$(5.11) \qquad I_i := \left(\frac{2i-2}{m}, \frac{2i+1}{m}\right) \cap [0,1], \quad i = 0, \ldots, N+1.$$

In this way $I_0 = [0, \frac{1}{m})$ and $I_{N+1} = (1 - \frac{1}{m}, 1]$. Note that

$$(5.12) \qquad I_i \cap I_j = \varnothing, \quad i, j = 0, \ldots, N+1, \ |i - j| > 1.$$

The interval I_i has length $\frac{3}{m}$ when $i = 1, \ldots, N$.

For $i = 1, \ldots, N$, define functions $\alpha_i : [0,1] \to [0,1]$ by:

$$(5.13) \qquad \alpha_i|_{[0,\frac{2i-2}{m}]} \equiv 0, \quad \alpha_i|_{[\frac{2i+1}{m},1]} \equiv 1, \quad \alpha_i|_{[\frac{2i-2}{m},\frac{2i+1}{m}]} \text{ is linear.}$$

Note that α_i maps I_i to $(0,1)$ by a linear homeomorphism, stretching by $\frac{m}{3}$. Hence, let us record for future use,

$$(5.14) \qquad (\tau_{\mathrm{Leb}} \otimes \tau_A)(x \circ \alpha_i) = \frac{3}{m}(\tau_{\mathrm{Leb}} \otimes \tau_A)(x), \quad x \in C_0((0,1), A).$$

We then define positive contractions in $C([0,1])$ by

$$(5.15) \qquad f_i := f \circ \alpha_i, \quad \grave{g}_i := \grave{g} \circ \alpha_i, \quad \acute{g}_i := \acute{g} \circ \alpha_i, \quad i = 1, \dots, N,$$

where $f, \grave{g}, \acute{g}$ are defined in (5.3). Set $\grave{g}_0 := \grave{g}_{-1} \circ \alpha_1$ and $\acute{g}_{N+1} := \acute{g}_{+1} \circ \alpha_N$. These definitions ensure that

$$(5.16) \qquad\qquad\qquad \grave{g}_{i-1} = \grave{g}_{-1} \circ \alpha_i, \quad \text{and}$$

$$(5.17) \qquad\qquad\qquad \acute{g}_{i+1} = \acute{g}_{+1} \circ \alpha_i, \quad i = 1, \dots, N.$$

Define f_0 and f_{N+1} by

$$f_0|_{[0,\frac{1}{3m}]} \equiv 1, \qquad\qquad f_0|_{[\frac{2}{3m},1]} \equiv 0,$$

$$(5.18) \qquad f_{N+1}|_{[0,1-\frac{2}{3m}]} \equiv 0, \qquad\qquad f_{N+1}|_{[1-\frac{1}{3m},1]} \equiv 1,$$

and again, demanding that f_0 and f_{N+1} are linear on the complements of the intervals of definition above. Note that

$$(5.19) \qquad\qquad\qquad f_i \in C_0(I_i), \quad i = 0, \dots, N+1.$$

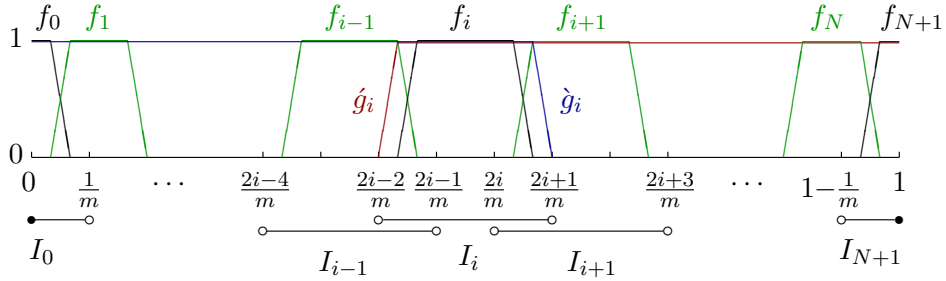These functions and intervals are displayed in Figure 4.

FIGURE 4. The intervals I_i and functions f_i, $\grave{g}_i$, $\acute{g}_i$.

By construction,

$$(5.20) \qquad\qquad\qquad \sum_{i=0}^{N+1} f_i = 1_{C([0,1])},$$

and in fact,

$$(5.21) \qquad\qquad (f_{i-1} + f_i + f_{i+1})|_{I_i} \equiv 1, \quad i = 0, \dots, N+1,$$

where we define $f_{-1} := 0 =: f_{N+2}$ to handle the end point cases. Also each $\grave{g}_i$ and $\acute{g}_i$ acts as a unit on the corresponding f_i, i.e.

$$f_i = \grave{g}_i f_i = f_i \grave{g}_i, \quad i = 0, \dots, N,$$

$$(5.22) \qquad f_i = \acute{g}_i f_i = f_i \acute{g}_i, \quad i = 1, \dots, N+1.$$

Further, for $i = 1, \ldots, N$,

$$(5.23) \qquad f_i \grave{g}_{i-1} \in C_0\left(\left(\tfrac{2i-2}{m}, \tfrac{2i-1}{m}\right)\right) \text{ and } \acute{g}_{i+1} f_i \in C_0\left(\left(\tfrac{2i}{m}, \tfrac{2i+1}{m}\right)\right).$$

For $i = 1, \ldots, N$, write

$$(5.24) \qquad \mathcal{F}_i := \{f_i \otimes a \mid a \in \mathcal{F}_A\} \cup \{f_i \otimes ab \mid a, b \in \mathcal{F}_A\} \quad \text{and}$$

$$(5.25) \qquad \mathcal{F}_i' := \{f_i, \grave{g}_i, \acute{g}_i, \grave{g}_{i-1}, \acute{g}_{i+1}\};$$

slightly abusing notation, by (5.4), (5.15), (5.16) and (5.17), we have

$$(5.26) \qquad \mathcal{F}_i = \mathcal{F} \circ \alpha_i \quad \text{and} \quad \mathcal{F}_i' = \mathcal{F}' \circ \alpha_i.$$

Use Lemma 2.6 to find a unital *-homomorphism $\Theta : C([0,1]) \to \mathcal{Q}_\omega$ and compatible *-homomorphisms $\grave{\Phi} : C_0([0,1), A) \to \mathcal{Q}_\omega$, $\acute{\Phi} : C_0((0,1], A) \to \mathcal{Q}_\omega$ satisfying

$$(5.27) \qquad \tau_{\mathcal{Q}_\omega} \circ \grave{\Phi} = \tau_{\mathrm{Leb}} \otimes \tau_A \quad \text{and}$$

$$(5.28) \qquad \tau_{\mathcal{Q}_\omega} \circ \acute{\Phi} = \tau_{\mathrm{Leb}} \otimes \tau_A.$$

Then define *-homomorphisms $\sigma_0, \ldots, \sigma_N$ by

$$\sigma_0 := \grave{\Phi}^{\oplus N} : C_0([0,1), A) \to \mathcal{Q}_\omega \otimes M_N,$$

$$\sigma_N := \acute{\Phi}^{\oplus N} : C_0((0,1], A) \to \mathcal{Q}_\omega \otimes M_N, \text{ and}$$

$$(5.29) \qquad \sigma_i := \grave{\Phi}|_{C_0((0,1),A)}^{\oplus(N-i)} \oplus \acute{\Phi}|_{C_0((0,1),A)}^{\oplus i} : C_0((0,1), A) \to \mathcal{Q}_\omega \otimes M_N$$

for $i = 1, \ldots, N-1$. Note that while $\sigma_1, \ldots, \sigma_{N-1}$ are defined on the suspension $C_0((0,1), A)$, σ_0 and σ_N are defined on the cones $C_0([0,1), A)$ and $C_0((0,1], A)$ respectively.

By applying Lemma 1.6 to the positive contraction $\Theta(\mathrm{id}_{[0,1]})$, and the intervals $I_0, \ldots, I_{N+1}$, there exist projections $q_0, \ldots, q_{N+1} \in \mathcal{Q}_\omega$ such that Property 5.3 below is satisfied. For condition (ii) below, the corresponding property in Lemma 1.6 ensures that q_i acts as a unit on $\Theta(C_0(I_i))$, and therefore on the hereditary C*-subalgebra generated by $\Theta(C_0(I_i))$, which contains $\grave{\Phi}(C_0(I_i), A)$ and $\acute{\Phi}(C_0(I_i), A)$ by compatibility. Condition (iii) below follows from the corresponding property in Lemma 1.6 and (5.21), while condition (iv) uses the corresponding property from Lemma 1.6 and (5.12).

Property 5.3. *(i) Each q_i commutes with the image of Θ.*
(ii) For each $i = 1, \ldots, N$, the projection q_i acts as a unit on $\Theta(C_0(I_i))$, $\grave{\Phi}(C_0(I_i), A))$ and $\acute{\Phi}(C_0(I_i), A))$, and hence $q_i^{\oplus N}$ acts as a unit on the image of $\sigma_j|_{C_0(I_i, A)}$ for all j. Likewise, q_0 acts as a unit on $\Theta(C_0(I_0))$ and $\grave{\Phi}(C_0(I_0), A))$, while q_{N+1} acts as a unit on both $\Theta(C_0(I_{N+1}))$ and $\acute{\Phi}(C_0(I_{N+1}), A))$.
(iii) For each $i = 0, \ldots, N+1$, $\Theta(f_{i-1} + f_i + f_{i+1})$ acts as a unit on q_i.
(iv) $q_i q_j = 0$ for $|i - j| > 1$.
(v)

$$(5.30) \qquad \tau_{\mathcal{Q}_\omega}(q_i) = |I_i| = \begin{cases} \frac{3}{m}, & i = 1, \ldots, N; \\ \frac{1}{m}, & i = 0 \text{ or } i = N+1. \end{cases}$$

We will now use the patching lemma of the previous section to find a family of approximately multiplicative, approximately compatible maps $\rho_0, \ldots, \rho_{N+1}$ such that ρ_i agrees with σ_{i-1} on the left-hand third of I_i, i.e., $(\frac{2i-2}{m}, \frac{2i-1}{m})$ and agrees with σ_i on the right-hand third of I_i, namely $(\frac{2i}{m}, \frac{2i+1}{m})$. Our precise objective is set out in the following claim.

Claim 5.4. *There exist completely positive maps*

$$(5.31) \qquad \rho_i : C_0(I_i, A) \to q_i \mathcal{Q}_\omega q_i \otimes M_N \otimes M_2, \quad i = 0, \ldots, N+1$$

such that:

(i) *For $i = 0, \ldots, N+1$, $(\tau_{\mathcal{Q}_\omega} \otimes \tau_{M_{2N}}) \circ \rho_i = \frac{1}{2} \tau_{\mathrm{Leb}} \otimes \tau_A$.*
(ii) *As maps into $\mathcal{Q}_\omega \otimes M_N \otimes M_2$, we have*

$$(5.32) \qquad \rho_0 = \sigma_0|_{C_0([0, \frac{1}{m}), A)} \oplus 0_N,$$

$$(5.33) \qquad \rho_{N+1} = \sigma_N|_{C_0((1-\frac{1}{m}, 1], A)} \oplus 0_N,$$

$$(5.34) \qquad \rho_i|_{C_0((\frac{2i-2}{m}, \frac{2i-1}{m}), A)} = \sigma_{i-1}|_{C_0((\frac{2i-2}{m}, \frac{2i-1}{m}), A)} \oplus 0_N,$$

$$(5.35) \qquad \rho_i|_{C_0((\frac{2i}{m}, \frac{2i+1}{m}), A)} = \sigma_i|_{C_0((\frac{2i}{m}, \frac{2i+1}{m}), A)} \oplus 0_N, \quad i = 1, \ldots, N.$$

(iii) *For $i = 1, \ldots, N$, $x, y \in \mathcal{F}_i$, and $h \in \mathcal{F}_i'$,*

$$(5.36) \qquad \rho_i(xy) \approx_\eta \rho_i(x)\rho_i(y),$$

$$(5.37) \qquad \rho_i(x)\Theta^{\oplus 2N}(h) \approx_\eta \rho_i(xh) \approx_\eta \Theta^{\oplus 2N}(h)\rho_i(x),$$

while ρ_0 and ρ_{N+1} are *-homomorphisms compatible with $\Theta^{\oplus 2N}$.

Proof of claim. The maps ρ_0 and ρ_{N+1} are already prescribed by (5.32) and (5.33); since $q_0^{\oplus N}$ and $q_{N+1}^{\oplus N}$ act as units on the images of $\sigma_0|_{C_0(I_0, A)}$ and $\sigma_{N+1}|_{C_0(I_{N+1}, A)}$ respectively by Property 5.3 (ii), ρ_0 and ρ_{N+1} have the codomains required by (5.31). Further, (5.32) and (5.33) ensure that ρ_0 and ρ_{N+1} are *-homomorphisms compatible with $\Theta^{\oplus 2N}$ which satisfy (i).

Fix $i \in \{1, \ldots, N\}$. Using the subalgebra C of $C([0,1], A)$ from (5.7), define $\phi, \psi : C \to q_i \mathcal{Q}_\omega q_i$ by

$$(5.38) \qquad \phi(c) := \lambda q_i + \grave{\Phi}(x \circ \alpha_i) \quad \text{and} \quad \psi(c) := \lambda q_i + \acute{\Phi}(x \circ \alpha_i)$$

for $c = x + \lambda 1_C$ where $x \in C_0((\frac{1}{3}, \frac{2}{3}), A)$. As $x \circ \alpha_i \in C_0(I_i)$, Property 5.3 (ii) shows that ϕ and ψ define unital *-homomorphisms.

We will take

$$(5.39) \qquad \iota := \begin{cases} \phi, & i \le n; \\ \psi, & i > n \end{cases}$$

in Property 5.2 so let us now show that both ϕ and ψ are Δ-full maps.

The unique normalised trace $\tau_{q_i \mathcal{Q}_\omega q_i}$ on $q_i \mathcal{Q}_\omega q_i$ is given by

$$(5.40) \qquad \tau_{q_i \mathcal{Q}_\omega q_i}(z) = \frac{1}{\tau_{\mathcal{Q}_\omega}(q_i)} \tau_{\mathcal{Q}_\omega}(z) \overset{(5.30)}{=} \frac{m}{3} \tau_{\mathcal{Q}_\omega}(z), \quad z \in q_i \mathcal{Q}_\omega q_i.$$

For $c = x + \lambda 1_C \in C_+\backslash\{0\}$ where $x \in C_0((\frac{1}{3}, \frac{2}{3}), A)$ and $\lambda \in \mathbb{C}$, we have

$$
\begin{aligned}
\tau_{q_i\mathcal{Q}_\omega q_i}(\phi(c)) &\overset{(5.38)}{=} \tau_{q_i\mathcal{Q}_\omega q_i}(\lambda 1_{q_i\mathcal{Q}_\omega q_i} + \grave{\Phi}(x \circ \alpha_i)) \\
&\overset{(5.40)}{=} \lambda + \frac{m}{3}\tau_{\mathcal{Q}_\omega}(\grave{\Phi}(x \circ \alpha_i)) \\
&\overset{(5.27)}{=} \lambda + \frac{m}{3}(\tau_{\text{Leb}} \otimes \tau_A)(x \circ \alpha_i) \\
&\overset{(5.14)}{=} \lambda + (\tau_{\text{Leb}} \otimes \tau_A)(x) \\
&\overset{(5.7)}{=} (\tau_{\text{Leb}} \otimes \tau_A)(c) \\
&\overset{(5.8)}{>} \frac{2}{\sqrt{\Delta(c)}}.
\end{aligned}
$$

(5.41)

Using (5.28) in place of (5.27), and otherwise computing identically, we also have

(5.42) $$\tau_{q_i\mathcal{Q}_\omega q_i}(\psi(c)) = (\tau_{\text{Leb}} \otimes \tau_A)(c) > \tfrac{2}{\sqrt{\Delta(c)}}.$$

Entering (5.41) and (5.42) into Lemma 2.2 (noting that $\sqrt{\Delta(c)} \in \mathbb{N}$) shows that ϕ and ψ are Δ-full.

Since $\grave{\Phi}$ is contractible, so is $\grave{\Phi}|_{C_0(I_i,A)} : C_0(I_i, A) \to \mathcal{Q}_\omega$. By Proposition 1.3 (ii), there exist $k \in \mathbb{N}$ and a (not necessarily unital) embedding $\mathcal{Q}_\omega \hookrightarrow q_i\mathcal{Q}_\omega q_i \otimes M_k$ which maps $q_i z q_i \in \mathcal{Q}_\omega$ to $q_i z q_i \otimes e_{11}$. Therefore, defining $\phi_0, \psi_0 : C_0((\frac{1}{3}, \frac{2}{3}), A) \to q_i\mathcal{Q}_\omega q_i \otimes M_k$ by

$$\phi_0(x) := \grave{\Phi}(x \circ \alpha_i) \otimes e_{11}, \quad x \in C_0((\tfrac{1}{3}, \tfrac{2}{3}), A),$$

(5.43) $$\psi_0(x) := \acute{\Phi}(x \circ \alpha_i) \otimes e_{11}, \quad x \in C_0((\tfrac{1}{3}, \tfrac{2}{3}), A),$$

ϕ_0 and ψ_0 are homotopic.[17] Thus the unitisations of ϕ_0 and ψ_0 are homotopic, hence induce the same map on $\underline{K}$, and $\phi_* = \psi_*$.[18]

By Property 5.2, there exists a unitary $u \in M_{n+1}(q_i\mathcal{Q}_\omega q_i)$ such that

(5.44) $$u(\phi(x) \oplus \iota^{\oplus n}(x))u^* \approx_\delta \psi(x) \oplus \iota^{\oplus n}(x), \quad x \in \mathcal{G}.$$

For $i \leq n$, this is

(5.45) $$u(\phi^{\oplus(n+1)}(x))u^* \approx_\delta \psi(x) \oplus \phi^{\oplus n}(x), \quad x \in \mathcal{G}.$$

Thus working in $q_i\mathcal{Q}_\omega q_i \otimes M_N$, we have

$$
\begin{aligned}
&(u\phi^{\oplus(n+1)}(x)u^*) \oplus \phi^{\oplus(N-i-n)}(x) \oplus \psi^{\oplus(i-1)}(x) \\
&\approx_\delta \psi(x) \oplus \phi^{\oplus n}(x) \oplus \phi^{\oplus(N-i-n)}(x) \oplus \psi^{\oplus(i-1)}(x), \quad x \in \mathcal{G}.
\end{aligned}
$$

(5.46)

Applying appropriate permutation unitaries to both sides of (5.46) we obtain a unitary $v \in q_i\mathcal{Q}_\omega q_i \otimes M_N$ with

(5.47) $$v(\phi^{\oplus(N-i+1)}(x) \oplus \psi^{\oplus(i-1)}(x))v^* \approx_\delta \phi^{\oplus(N-i)}(x) \oplus \psi^{\oplus i}(x), \quad x \in \mathcal{G}.$$

---

[17]The point here is that while $\acute{\Phi}$ and $\grave{\Phi}$ restrict to contractible and hence homotopic maps $C_0(I_i, A) \to \mathcal{Q}_\omega$, it doesn't simply follow that these restrictions are homotopic within the space of *-homomorphisms $C_0(I_i, A) \to q_i\mathcal{Q}_\omega q_i$.

[18]The maps ϕ_0 and ψ_0 have unitisations $\phi \oplus \pi^{\oplus(k-1)}$ and $\psi \oplus \pi^{\oplus(k-1)}$ respectively, where $\pi : C \to q_i\mathcal{Q}_\omega q_i$ is the canonical unital *-homomorphism $C \to q_i\mathcal{Q}_\omega q_i$ with kernel $C_0((\frac{1}{3}, \frac{2}{3}), A)$. Thus $\phi_* + (k-1)\pi_* = \psi_* + (k-1)\pi_*$ and subtracting $(k-1)\pi_*$ from both sides gives $\phi_* = \psi_*$.

When $i > n$, we have $\iota = \psi$, so that adding $\phi^{\oplus(N-i)}(x) \oplus \psi^{\oplus(i-n-1)}(x)$ to both sides of (5.44) and applying appropriate permutation unitaries in $q_i \mathcal{Q}_\omega q_i \otimes M_N$ likewise gives a unitary $v \in q_i \mathcal{Q}_\omega q_i \otimes M_N$ satisfying (5.47).

We next wish to use Property 5.1, taking $E := q_i \mathcal{Q}_\omega q_i \otimes M_N$. To this end, we define maps $\nu_0, \nu_1 : C_0((0,1), A) \to q_i \mathcal{Q}_\omega q_i \otimes M_N$ by

$$(5.48) \qquad \nu_j(x) := \sigma_{i+j-1}(x \circ \alpha_i), \quad x \in C_0((0,1), A), \; j = 0, 1,$$

noting that the codomain can indeed be taken to be $q_i \mathcal{Q}_\omega q_i \otimes M_N$ as α_i maps I_i homeomorphically onto $(0,1)$ by (5.13), and $q_i^{\oplus N}$ acts as a unit on the image of $\sigma_{i-1}|_{C_0(I_i, A)}$ and $\sigma_i|_{C_0(I_i, A)}$ by Property 5.3 (ii). By construction

$$(5.49) \qquad \nu_0|_{C_0((\frac{1}{3}, \frac{2}{3}), A)} = (\phi^{\oplus(N-i+1)} \oplus \psi^{\oplus(i-1)})|_{C_0((\frac{1}{3}, \frac{2}{3}), A)},$$

$$(5.50) \qquad \nu_1|_{C_0((\frac{1}{3}, \frac{2}{3}), A)} = (\phi^{\oplus(N-i)} \oplus \psi^{\oplus i})|_{C_0((\frac{1}{3}, \frac{2}{3}), A)}.$$

Since $\mathcal{G} \subset C_0((\frac{1}{3}, \frac{2}{3}), A)$, it follows that the unitary v giving (5.47) verifies (5.6) of Property 5.1.

Define $\theta : C([0,1]) \to q_i \mathcal{Q}_\omega q_i \otimes M_N$ by

$$(5.51) \qquad \theta(h) := (q_i \Theta(h \circ \alpha_i) q_i)^{\oplus N}, \quad h \in C([0,1]);$$

this is a unital *-homomorphism by Property 5.3 (i). Since $\acute{\Phi}, \grave{\Phi}$ are compatible with Θ, the definitions in (5.29) and (5.48) ensure that ν_0 and ν_1 are compatible with θ. For $x \in C_0((0,1), A)$ and $j = 0, 1$, we compute

$$(\tau_{q_i \mathcal{Q}_\omega q_i} \otimes \tau_{M_N})(\nu_j(x)) \overset{(5.40),(5.48)}{=} \frac{m}{3}(\tau_{\mathcal{Q}_\omega} \otimes \tau_{M_N})(\sigma_{i+j-1}(x \circ \alpha_i))$$

$$\overset{(5.29)}{=} \frac{m}{3N} \tau_{\mathcal{Q}_\omega}\big((N-i-j+1)\grave{\Phi}(x \circ \alpha_i)$$
$$+ (i+j-1)\acute{\Phi}(x \circ \alpha_i)\big)$$

$$\overset{(5.27),(5.28)}{=} \frac{m}{3N} N(\tau_{\mathrm{Leb}} \otimes \tau_A)(x \circ \alpha_i)$$

$$(5.52) \qquad\qquad\qquad \overset{(5.14)}{=} (\tau_{\mathrm{Leb}} \otimes \tau_A)(x).$$

This completes the verification of the hypotheses in Property 5.1, and so we obtain a completely positive map

$$(5.53) \qquad \rho : C_0((0,1), A) \to q_i \mathcal{Q}_\omega q_i \otimes M_N \otimes M_2 = E \otimes M_2$$

satisfying (i)-(v) of Property 5.1. Define a completely positive map $\rho_i : C_0(I_i, A) \to q_i \mathcal{Q}_\omega q_i \otimes M_N \otimes M_2$ by

$$(5.54) \qquad \rho_i(x) := \rho(x \circ \alpha_i|_{I_i}^{-1}), \quad x \in C_0(I_i, A),$$

(which makes sense as $\alpha_i|_{I_i} : I_i \to (0,1)$ is a homeomorphism).

To end the proof of the claim, we link Property 5.1 to the conditions of Claim 5.4. Property 5.1 (iii), (5.14), and (5.40) combine to give Claim 5.4 (i), while (5.48), Property 5.1 (i) and (ii) give (5.34) and (5.35) respectively. Using (5.26), (5.48), and (5.51), the estimates for ρ in Properties 5.1 (iv), (v) in terms of $\mathcal{F}$ and $\mathcal{F}'$ are transformed to give the required estimates (5.36), (5.37) for the sets $\mathcal{F}_i$ and $\mathcal{F}'_i$. This concludes the proof of the claim. $\qquad \square$

We now define the completely positive map

(5.55) $$\Psi : A \to \mathcal{Q}_\omega \otimes M_N \otimes M_2$$

to be used to witness quasidiagonality of τ_A (via (5.1) and (5.2)) by

(5.56) $$\Psi(a) := \sum_{i=0}^{N+1} \rho_i(f_i \otimes a), \quad a \in A.$$

Note that $\rho_i(f_i \otimes a)$ makes sense since $f_i \in C_0(I_i)$. Condition (i) of Claim 5.4 gives $(\tau_{Q_\omega} \otimes \tau_{M_{2N}})(\rho_i(f_i \otimes a)) = \frac{1}{2}\tau_{\text{Leb}}(f_i)\tau_A(a)$, for each i and $a \in A$. Then, as the f_i form a partition of unity for $C([0,1])$ by (5.20),

$$(\tau_{\mathcal{Q}_\omega} \otimes \tau_{M_{2N}})(\Psi(a)) = \sum_{i=0}^{N+1} (\tau_{\mathcal{Q}_\omega} \otimes \tau_{M_{2N}})\rho_i(f_i \otimes a)$$

$$= \frac{1}{2} \sum_{i=0}^{N+1} \tau_{\text{Leb}}(f_i)\tau_A(a)$$

(5.57) $$= \frac{1}{2}\tau_A(a), \quad a \in A,$$

establishing (5.2).

We complete the proof by showing (5.1), so fix $a, b \in \mathcal{F}_A$. By Property 5.3 (iv) $q_i q_j = 0$ when $|i - j| > 1$, and thus the images of ρ_i and ρ_j are orthogonal for $|i - j| > 1$. For notational simplicity, define both ρ_{-1}, ρ_{N+2} to be the zero map and recall the similar convention $f_{-1} = 0 = f_{N+2}$. Then we have

$$\Psi(a)\Psi(b) = \sum_{i,j=0}^{N+1} \rho_i(f_i \otimes a)\rho_j(f_j \otimes b)$$

(5.58) $$= \sum_{k=-1}^{1} \sum_{i=0}^{N+1} \rho_i(f_i \otimes a)\rho_{i+k}(f_{i+k} \otimes b).$$

By Property 5.3 (iii) and (5.31), we also have

(5.59) $$\Psi(ab) = \sum_{k=-1}^{1} \sum_{i=0}^{N+1} \rho_i(f_i \otimes ab)\Theta^{\oplus 2N}(f_{i+k}).$$

In this way, we can write

$$\Psi(a)\Psi(b) - \Psi(ab)$$

$$= \sum_{k=-1}^{1} \sum_{\substack{i=0 \\ i \text{ odd}}}^{N+1} \rho_i(f_i \otimes a)\rho_{i+k}(f_{i+k} \otimes b) - \rho_i(f_i \otimes ab)\Theta^{\oplus 2N}(f_{i+k})$$

(5.60) $$+ \sum_{k=-1}^{1} \sum_{\substack{i=0 \\ i \text{ even}}}^{N+1} \rho_i(f_i \otimes a)\rho_{i+k}(f_{i+k} \otimes b) - \rho_i(f_i \otimes ab)\Theta^{\oplus 2N}(f_{i+k}).$$

The decomposition into odd and even i in the sums in (5.60) gives the estimate

$$\|\Psi(a)\Psi(b) - \Psi(ab)\|$$

$$(5.61) \quad \leq 2\sum_{k=-1}^{1} \max_i \|\rho_i(f_i \otimes a)\rho_{i+k}(f_{i+k} \otimes b) - \rho_i(f_i \otimes ab)\Theta^{\oplus 2N}(f_{i+k})\|.$$

This follows as, for k fixed, writing

$$(5.62) \qquad T_{i,k} := \rho_i(f_i \otimes a)\rho_{i+k}(f_{i+k} \otimes b) - \rho_i(f_i \otimes ab)\Theta^{\oplus 2N}(f_{i+k}),$$

we have $T_{i,k} = q_i^{\oplus 2N} T_{i,k} q_{i+k}^{\oplus 2N}$ (recall that q_{i+k} acts as a unit on $\Theta(f_{i+k})$). As $(q_i)_{i \text{ even}}$ are pairwise orthogonal projections and $(q_{i+k})_{i \text{ even}}$ are also pairwise orthogonal projections[19] one has $\|\sum_{i \text{ even}} T_{i,k}\| = \max\{\|T_{i,k}\| \mid i \text{ even}\}$. Likewise $\|\sum_{i \text{ odd}} T_{i,k}\| = \max\{\|T_{i,k}\| \mid i \text{ odd}\}$, giving (5.61).

We now estimate the terms in (5.61), starting with the case $k = 0$. For $i = 1, \ldots, N$ we have $f_i \otimes a, f_i \otimes b, f_i \otimes ab \in \mathcal{F}_i$ by (5.24) and $f_i \in \mathcal{F}_i'$ by (5.25), so that

$$(5.63) \qquad \rho_i(f_i \otimes a)\rho_i(f_i \otimes b) \stackrel{(5.36)}{\approx_\eta} \rho_i(f_i^2 \otimes ab)$$

$$\stackrel{(5.37)}{\approx_\eta} \rho_i(f_i \otimes ab)\Theta^{\oplus 2N}(f_i).$$

For $i = 0$ or $i = N+1$, one has $\rho_i(f_i \otimes a)\rho_i(f_i \otimes b) = \rho_i(f_i \otimes ab)\Theta^{\oplus 2N}(f_i)$, as ρ_0 and ρ_{N+1} are homomorphisms compatible with $\Theta^{\oplus 2N}$. Thus the $k = 0$ contribution to (5.61) is at most 4η.

Now fix $k = \pm 1$, and $i = 0, \ldots, N+1$ such that $i + k \in \{0, \ldots, N+1\}$. Set

$$(5.64) \qquad \tilde{g}_i := \begin{cases} \acute{g}_i, & k = -1; \\ \grave{g}_i, & k = +1, \end{cases} \quad \text{and} \quad \tilde{g}_{i+k} := \begin{cases} \grave{g}_{i+k}, & k = -1; \\ \acute{g}_{i+k}, & k = +1. \end{cases}$$

In this way, (5.22) (see Figure 4) gives

$$(5.65) \qquad \tilde{g}_i f_i = f_i = f_i \tilde{g}_i \quad \text{and} \quad \tilde{g}_{i+k} f_{i+k} = f_{i+k} = f_{i+k}\tilde{g}_{i+k},$$

so that as $\tilde{g}_i, \tilde{g}_{i+k} \in \mathcal{F}_i'$ and $\tilde{g}_i, \tilde{g}_{i+k} \in \mathcal{F}_{i+k}'$ (see (5.25)), we have[20]

$$\rho_i(f_i \otimes a)\rho_{i+k}(f_{i+k} \otimes b)$$

$$\stackrel{(5.65)}{=} \rho_i(f_i\tilde{g}_i \otimes a)\rho_{i+k}(\tilde{g}_{i+k}f_{i+k} \otimes b)$$

$$\stackrel{(5.37)}{\approx_{2\eta}} \rho_i(f_i \otimes a)\Theta^{\oplus 2N}(\tilde{g}_i\tilde{g}_{i+k})\rho_{i+k}(f_{i+k} \otimes b)$$

$$(5.66) \qquad \stackrel{(5.37)}{\approx_{2\eta}} \rho_i(f_i\tilde{g}_{i+k} \otimes a)\rho_{i+k}(\tilde{g}_i f_{i+k} \otimes b).$$

Set

$$(5.67) \qquad \qquad \tilde{\sigma}_i := \begin{cases} \sigma_{i-1}, & k = -1; \\ \sigma_i, & k = +1. \end{cases}$$

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[19]Though not necessarily pairwise orthogonal to $(q_i)_{i \text{ even}}$.

[20]When one of i or $i+k$ is 0 or $N+1$, exact compatibility of ρ_0 and ρ_{N+1} can be used in place of (5.37), leading to better estimates.

Then, using (5.23), (5.34) and (5.35) for the first and last equalities below, we have

$$\rho_i(f_i\tilde{g}_{i+k}\otimes a)\rho_{i+k}(\tilde{g}_i f_{i+k}\otimes b)$$
$$= \big(\tilde{\sigma}_i(f_i\tilde{g}_{i+k}\otimes a)\oplus 0_N\big)\big(\tilde{\sigma}_i(\tilde{g}_i f_{i+k}\otimes b)\oplus 0_N\big)$$
$$\overset{\text{Compatibility, (5.65)}}{=} \big(\tilde{\sigma}_i(f_i\tilde{g}_{i+k}\otimes ab)\oplus 0_N\big)\Theta^{\oplus 2N}(f_{i+k})$$
$$(5.68)\qquad = \rho_i(f_i\tilde{g}_{i+k}\otimes ab)\Theta^{\oplus 2N}(f_{i+k}).$$

Finally we use that $\tilde{g}_{i+k}\in\mathcal{F}'_i$ again[21] to see that

$$\rho_i(f_i\tilde{g}_{i+k}\otimes ab)\Theta^{\oplus 2N}(f_{i+k})$$
$$\overset{(5.37)}{\approx_\eta} \rho_i(f_i\otimes ab)\Theta^{\oplus 2N}(f_{i+k}\tilde{g}_{i+k})$$
$$(5.69)\qquad \overset{(5.65)}{=} \rho_i(f_i\otimes ab)\Theta^{\oplus 2N}(f_{i+k}).$$

Combining (5.66), (5.68) and (5.69) gives

$$(5.70)\qquad \rho_i(f_i\otimes a)\rho_{i+k}(f_{i+k}\otimes b)\approx_{5\eta}\rho_i(f_i\otimes ab)\Theta^{\oplus 2N}(f_{i+k}),$$

so that the $k=\pm 1$ terms in (5.61) contribute at most 10η each. Putting these estimates together with the $k=0$ case, (5.61) gives

$$(5.71)\qquad \|\Psi(ab)-\Psi(a)\Psi(b)\|\leq 24\eta\overset{(5.5)}{<}\varepsilon,\quad a,b\in\mathcal{F}_A,$$

establishing (5.2) and hence completing the proof of Theorem A. $\qquad\square$

Remark 5.5. It is worth noting that in the argument above we can replace $\mathcal{Q}_\omega$ by the ultrapower of the CAR algebra[22] M_{2^∞} to produce approximately multiplicative maps $\Psi:A\to(M_{2^\infty})_\omega\otimes M_{2N}$ which realise $\frac{1}{2}\tau_A$. In the proof, we can also arrange for N to be of the form 2^k, so that $(M_{2^\infty})_\omega\otimes M_{2N}\cong(M_{2^\infty})_\omega$, whence reindexing will yield a *-homomorphism $\Phi':A\to(M_{2^\infty})_\omega$ realising $\frac{1}{2}\tau_A$. This will induce (as τ_A is faithful) a unital embedding of A into $\Theta(1_A)(M_{2^\infty})_\omega\Theta(1_A)$, which is isomorphic to an ultraproduct of UHF algebras. Note, however, that this does not necessarily give a unital embedding of A into $(M_{2^\infty})_\omega$. Indeed, the unit of A might be exactly 3-divisible in K_0, whence no such embedding exists.

Remark 5.6. We need to assume that τ_A is faithful since – unlike nuclearity – the property of satisfying the UCT does not in general pass to quotients.[23] However, we will see in Corollary 6.1 below that in the situation of the theorem (when A has a faithful trace to begin with), all traces, faithful or not, are quasidiagonal.

In order to facilitate a forthcoming generalisation by Jamie Gabe of Theorem A to amenable traces on exact algebras by, we note that the above proof establishes the following statement.

---

[21]When $i=0$ or $i=N+1$, one has equality in the next estimate.

[22]Or any other ultraproduct of UHF algebras, since these are admissible target algebras of finite type.

[23]Every C*-algebra is the continuous image of its cone, which is homotopic to 0, so satisfies the UCT, and there exist non-nuclear C*-algebras which do not satisfy the UCT [64].

Lemma 5.7. *Let A be a separable unital exact C^*-algebra in the UCT class and let τ_A be a faithful trace on A. Suppose there are $*$-homomorphisms $\Theta : C([0,1]) \to \mathcal{Q}_\omega$, $\grave{\Phi} : C_0([0,1), A) \to \mathcal{Q}_\omega$ and $\acute{\Phi} : C_0((0,1], A) \to \mathcal{Q}_\omega$ such that Θ is unital, $\grave{\Phi}$ and $\acute{\Phi}$ are nuclear and compatible with Θ,*

$$(5.72) \qquad \tau_{\mathcal{Q}_\omega} \circ \grave{\Phi} = \tau_{\mathrm{Leb}} \otimes \tau_A, \quad and \quad \tau_{\mathcal{Q}_\omega} \circ \acute{\Phi} = \tau_{\mathrm{Leb}} \otimes \tau_A.$$

Then for any finite subset $\mathcal{F}_A$ of A and $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and a nuclear c.p. map $\Psi : A \to \mathcal{Q}_\omega \otimes M_{2N}$ such that (5.1) and (5.2) hold.

To see this, one works through the proof of Theorem A, checking that the hypothesis of Lemma 5.7 enables all maps in the proof be taken to be nuclear. Property 5.1 is obtained from the patching lemma (Lemma 4.1). By construction the ρ given by Lemma 4.1 is defined explicitly in terms of ν_0, ν_1 and a unitary u in (4.12), (4.13) and (4.15), and so is nuclear when ν_0 and ν_1 are. The controlled stable uniqueness theorem (Theorem 3.5) is valid for separable unital exact domains in the UCT class and nuclear $*$-homomorphisms, so Property 5.2 holds with ϕ and ψ additionally assumed nuclear. With these adjustments to Properties 5.1 and 5.2, the maps $\rho_0, \ldots, \rho_{N+1}$ produced in Claim 5.4 are all nuclear. The definition of Ψ in terms of these ρ_i from (5.56) ensures that it too is nuclear.

6. CONSEQUENCES: STRUCTURE AND CLASSIFICATION

In this section we shall explain how Corollaries B-E follow from Theorem A, and put into context the main result and its consequences for the structure and classification of simple nuclear C^*-algebras.

We begin with Corollary B. The version below includes an additional statement pointed out to us by Nate Brown; it implies that if there is a faithful (hence quasidiagonal) trace to begin with, then in fact all traces are quasidiagonal.

Corollary 6.1. *Every separable, nuclear C^*-algebra in the UCT class with a faithful trace is quasidiagonal. In particular, each simple, separable, stably finite nuclear C^*-algebra satisfying the UCT is quasidiagonal.*

Further if A is a separable, unital, nuclear and quasidiagonal C^*-algebra satisfying the UCT, then all traces on A are quasidiagonal.

Proof. The first sentence is an immediate consequence of Theorem A and the fact that a C^*-algebra with a faithful quasidiagonal trace is quasidiagonal (see Proposition 1.4 and Remark 1.5 (iii)).

For the second sentence, let A be a separable, simple, nuclear, stably finite C^*-algebra that satisfies the UCT. It suffices to show that there is a C^*-algebra stably isomorphic to A which has a trace (as such a trace is automatically faithful by simplicity). If $A \otimes \mathcal{K}$ contains a nonzero projection p, then the hereditary C^*-subalgebra $p(A \otimes \mathcal{K})p$ is stably isomorphic to A by L. Brown's Theorem ([7]), and has a trace by [4] and [33]. On the other hand, if A is stably projectionless then by [66, Corollary 2.2], A is stably isomorphic to an algebraically simple and stably finite C^*-algebra B. By [39] and [66, Corollary 2.5], B has a faithful trace, as required.

For the last claim, let $\mathcal{N}$ be the class of separable nuclear C*-algebras satisfying the UCT. Then $\mathcal{N}$ contains $\mathbb{C}$ and is closed under: countable inductive limits with injective connecting maps, tensoring by finite dimensional matrix algebras, and extensions[24] (for nuclearity this is a standard exercise, while for the UCT, this is a consequence of the characterisation in terms of the bootstrap class; see [3, Definition 22.3.4]). If $A \in \mathcal{N}$ is simple, then Theorem A shows that all traces on A are quasidiagonal. Thus [10, Lemma 6.1.20 (3) $\Rightarrow$ (1)] shows[25] that every quasidiagonal C*-algebra in $\mathcal{N}$ has the property that all of its amenable traces are quasidiagonal. $\qquad\square$

When G is countable, Corollary C follows from Theorem A, since $C_r^*(G)$ satisfies the UCT by a result of Tu [70, Lemma 3.5 and Proposition 10.7], and the canonical trace is well-known to be faithful. The general case holds since any discrete amenable group can be exhausted by an increasing net of countable amenable subgroups, and the same goes for the group C*-algebras; moreover, by Arveson's extension theorem, Voiculescu's approximation form of quasidiagonality is a local property and therefore follows from the separable case (see also the appendix of [9]).

The theorem below summarises a number of known facts (many of which are quite deep) and combines them with our main result.[26] Corollary D is obviously contained in Theorem 6.2 (iv). Although Corollary E can be read off from Theorem 6.2 (iv) in conjunction with [62], it is worth mentioning explicitly how it follows from published results together with Theorem A. In the traceless case, Corollary E is Kirchberg–Phillips classification (cf. [58, Theorem 8.4.1]) and doesn't use Theorem A. The monotracial situation follows from [46, Theorem 5.4] (which generalises [77]); since Theorem A makes quasidiagonality automatic, the tracial rank hypothesis required in [46] follows from [49, Theorem 6.1]. In both cases the Elliott invariant reduces to ordered K-theory.

Theorem 6.2. *Let A be a separable, simple, unital C*-algebra with finite nuclear dimension. Suppose in addition that A satisfies the UCT.[27] Then:*

(i) *A has nuclear dimension at most 2; if A has no traces then it has nuclear dimension 1, and if A has traces it has decomposition rank at most 2. If $T(A)$ is nonempty and has compact extreme boundary, then A in fact has decomposition rank 1 or 0, and the latter happens if and only if A is AF.*

(ii) *A is purely infinite if and only if A has no traces if and only if A has infinite decomposition rank.*

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[24]If $0 \to I \to E \to B \to 0$ is a short exact sequence with $I, B \in \mathcal{N}$, then $E \in \mathcal{N}$.

[25]The definition of a Popa algebra from [10, Definition 1.2] is not required here, all that matters is that such algebras are simple.

[26]In the case of at most one trace, Theorem 6.2 can be entirely derived from Theorem A in conjunction with published results, whereas some of the statements in the general form also employ arXiv preprints still under review.

[27]The UCT is not needed for some parts of the theorem. We make it a global assumption for the sake of brevity, but will flag up in the proof where it is actually used.

> *In this case, A is an inductive limit of direct sums of* C**-algebras of the form $\mathcal{O}_n \otimes M_k \otimes C(S^1)$, where $\mathcal{O}_n$ denotes a Cuntz algebra (including $n = \infty$).*
>
> (iii) *A is stably finite if and only if A has a trace if and only if the decomposition rank of A is finite.*
>   *In this case, A is an inductive limit of subhomogeneous* C**-algebras of topological dimension at most 2.*
>
> (iv) *If B is another such* C**-algebra (and both A and B are infinite dimensional), then any isomorphism between the Elliott invariants of A and B lifts to a *-isomorphism between the* C**-algebras.*

Proof. Let us begin with (iii). It is a well-known consequence of [4] and [33] that if A is stably finite then it has a trace;[28] the converse is trivial (and no use of the UCT is required for either direction).

If A has traces, these are all quasidiagonal by Theorem A and A is classifiable by [29, Theorem 4.3] (see also [29, Corollary 4.4 and Theorem 4.5]), via the main result of [32]. Now A is isomorphic to one of the models constructed in [28] (since those exhaust the invariant), which in turn have topological dimension at most 2, as pointed out in [74, 1.11]; the UCT is of course necessary for this argument.

By [74] these models have decomposition rank at most 2, and therefore so does the limit A by [42, (3.2)]. Conversely, finite decomposition rank implies quasidiagonality by [42, Theorem 4.4] and hence the existence of a trace by [72, 2.4] (neither of these implications require the UCT). This completes the proof of (iii).

We next turn to (ii). If A is purely infinite then it is clearly traceless; the converse follows from the dichotomy of [79, Theorem 5.4] which shows that A is either stably finite or purely infinite, the former being impossible by (iii) (neither direction requires the UCT). We already know from (iii) that A is traceless if and only if it has infinite decomposition rank; the reverse direction uses the UCT. The statement about inductive limits is [58, Corollary 8.4.11] and again requires the UCT.

For (i), the statement about AF algebras is [79, Remark 2.2 (iii)]; the UCT is not involved. In the traceless case A is a Kirchberg algebra by (ii) and we have nuclear dimension 1 by [6, Theorem G], still without assuming UCT. If A has a trace, we have already seen in (iii) that it has decomposition rank at most 2; the UCT is heavily involved. Under the extra trace space condition, the decomposition rank is at most 1 by [6, Theorem F], and the UCT still enters through Theorem A.

We have already mentioned that the classification statement of (iv) in the absence of traces is Kirchberg–Phillips classification (cf. [58, Theorem 8.4.1]). When there are traces, these are all quasidiagonal by Theorem A and the result follows as in (i) from [29, Theorem 4.3]. All of these require the UCT.   □

Theorem 6.2 mostly focuses on the classification of simple nuclear C*-algebras. For completeness we briefly mention the Toms–Winter conjecture

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[28]As A has finite nuclear dimension, the shorter argument of [12] can be used in place of [33].

which emphasises their structure and predicts that finite noncommutative topological dimension, $\mathcal{Z}$-stability, and strict comparison occur simultaneously. The conjecture makes sense for both nuclear dimension and decomposition rank acting as topological dimension; in the latter case it subsumes the Blackadar–Kirchberg problem (in the simple, unital case), whereas the nuclear dimension version encapsulates both the stably finite and the purely infinite situation. Combining the two settings leads to the full Toms–Winter conjecture for simple unital C*-algebras (see [69, Remark 3.5] and [79, Conjecture 9.3]).

Conjecture 6.3. *Let A be a separable, simple, unital, nuclear and infinite dimensional C*-algebra. Then, the following are equivalent.*

(i) A has finite nuclear dimension.
(ii) A is $\mathcal{Z}$-stable.
(iii) A has strict comparison.

If A is stably finite, (i) may be replaced by

(i') A has finite decomposition rank.

The implications (i') $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold in full generality (the first is trivial, the second is [76, Corollary 6.3], and the third is [59, Theorem 4.5]). When the extreme boundary of $T(A)$ is compact and finite dimensional,[29] we have (iii) $\Rightarrow$ (ii) by [41, 61, 67] (building on the unique trace case of [48]) and (ii) $\Rightarrow$ (i) is [6, Theorem B] when A has compact tracial extreme boundary; both conditions are of course satisfied in the monotracial case. All of these implications are independent of classification (or the UCT, for that matter). Theorem 6.2 yields (i) $\Rightarrow$ (i') for stably finite C*-algebras when they in addition satisfy the UCT. This is the first abstract result of this kind (although quasidiagonality was identified as the differentiating feature between nuclear dimension and decomposition rank in [49], particularly in light of [62]). The UCT enters through our main theorem and also through the full-blown classification result from [29]. In the case of compact tracial extreme boundary, (i) $\Rightarrow$ (i') follows from Theorem A together with [6, Theorem F]; when A is monotracial one can combine Theorem A with [49, Corollary 1.2] (still using the UCT and stable uniqueness, but not the full-fledged classification of simple C*-algebras). Upon taking the intersection of these conditions, we see that Conjecture 6.3 holds when A has compact (possibly empty) tracial extreme boundary with finite covering dimension. As a structural counterpart of Corollary E we highlight the special case of at most one trace (which again only involves our main theorem in conjunction with published results):

Corollary 6.4. *The full Toms–Winter conjecture (Conjecture 6.3) holds for C*-algebras with at most one trace and which satisfy the UCT.*

We close with some more concrete applications of classification methods. Recall that a C*-algebra is AF-embeddable if it is isomorphic to a C*-subalgebra of an AF algebra. One can refine this notion by asking for embeddings into simple AF algebras, or by prescribing a trace and asking this to be picked up by the embedding composed with a (perhaps even

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[29]Certain non-compact but still finite dimensional boundaries are handled in [80].

unique) trace on the AF algebra. AF-embeddability clearly implies quasidiagonality, and conversely many quasidiagonal algebras are known to be AF-embeddable, a standout result being Ozawa's homotopy invariance of AF-embeddability [51] (see [11, Chapter 8] for an overview and further results). As with Theorem 6.2, Theorem A can be used to remove quasidiagonality assumptions from applications of classification to AF-embeddability; we highlight the monotracial case.

Corollary 6.5. *Let A be a separable, simple, unital, monotracial and nuclear C^*-algebra which satisfies the UCT. Then A embeds unitally into a simple, monotracial AF algebra.*

Proof. Upon tensoring by the universal UHF algebra $\mathcal{Q}$, we may assume that A is $\mathcal{Q}$-stable. Then Theorem A provides the quasidiagonality required to use [49, Theorem 6.1] to see that A is TAF. The ordered K_0-group of A has Riesz interpolation and is simple and weakly unperforated (see [58, p. 59]); since $K_0(A)$ is torsion-free (A absorbs $\mathcal{Q}$), it is in fact unperforated (cf. [58, Definitions 1.4.3 and 3.3.2]) and hence a dimension group by [25]. Now by [58, Proposition 1.4.2 and Corollary 1.5.4] it is the ordered K_0-group of a unital, simple AF algebra B, which will automatically be $\mathcal{Q}$-stable (by Elliott's AF classification [27]) since $K_0(B) \cong K_0(B \otimes \mathcal{Q})$. Upon sending $K_1(A)$ to $\{0\} = K_1(B)$, we obtain a morphism between the ordered K-groups of the TAF algebras A and B, which lifts to a unital *-homomorphism from A to B by [20] (note again that by $\mathcal{Q}$-stability there is no torsion and so total K-theory reduces to just K-theory).[30] Since A is simple, this is automatically an embedding. Finally, since A is monotracial there is only one state on $K_0(A)$ (see [58, Theorem 1.1.11]), hence also on $K_0(B)$, which in turn implies (by [58, Proposition 1.1.12]) that B is monotracial as well. Since the embedding of A into B is unital, it necessarily preserves the trace. $\square$

In particular one can apply the previous corollary to enhance Corollary C, extending the corresponding result for elementary amenable, countable, discrete groups from [53].

Corollary 6.6. *If G is a countable, discrete, amenable group, then $C_r^*(G)$ embeds unitally into a unital, simple, monotracial AF algebra in such a way that the canonical trace on $C_r^*(G)$ is realised.*

Proof. By [53, Proposition 2.1], $C_r^*(G)$ embeds into a separable, unital, simple, nuclear, monotracial, UCT C^*-algebra $B(G)$ in a fashion which picks up the canonical trace on $C_r^*(G)$.[31] The result then follows from Corollary 6.5. $\square$

Strongly self-absorbing C^*-algebras, whose study from an abstract perspective was initiated in [68], are omnipresent in the structure and classification theory of nuclear C^*-algebras. These are simple, nuclear, with at most one trace (results that go back to [26]) and are $\mathcal{Z}$-stable by [75]. The

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[30]Lifting morphisms as opposed to isomorphisms is a special feature in classification which has been carried out for Kirchberg algebras and for TAF algebras, but remains yet to be carried out in a more general setting.

[31]$B(G)$ is the Bernoulli shift crossed product $\bigotimes_G (M_{2^\infty}) \rtimes G$, from which the statement about the preservation of the canonical trace follows.

only known examples are the Jiang–Su algebra $\mathcal{Z}$, UHF algebras of infinite type, the Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_2$, and tensor products of $\mathcal{O}_\infty$ with UHF algebras of infinite type. We are very interested in whether there are any other examples (cf. [68, Question 5.11]), mostly because strongly self-absorbing C*-algebras may be thought of as a microcosm of simple nuclear C*-algebras, and they witness important structural questions such as the UCT problem and the Blackadar–Kirchberg problem. When assuming the UCT we now have an answer, since in this case the possible K-groups were computed in [68] and we may apply Corollary E:

Corollary 6.7. *The strongly self-absorbing C*-algebras in the UCT class are precisely the known ones.*

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