The Cuntz semigroup of $C(X, A)$

Aaron Tikuisis
aptikuis@math.toronto.edu

University of Toronto

SGS Thesis Exam

The Cuntz semigroup

Ordered semigroup, constructed as follows.

- For $a, b \in (A \otimes \mathcal{K})_+$, $[a] \leq [b]$ if

$$\|a - s_n^* b s_n\| \to 0,$$

  for some $(s_n) \subset A \otimes \mathcal{K}$.
- $[a] = [b]$ if $[a] \leq [b]$ and $[b] \leq [a]$.
- $\mathcal{C}u(A) = \{[a] : a \in (A \otimes \mathcal{K})_+\}$
- $[a] + [b] := [a' + b']$ where $[a] = [a'], [b] = [b']$ and $a' \perp b'$.

The Cuntz semigroup

Ordered semigroup, constructed as follows.

- For $a, b \in (A \otimes \mathcal{K})_+$, $[a] \leq [b]$ if

$$\|a - s_n^* b s_n\| \to 0,$$

  for some $(s_n) \subset A \otimes \mathcal{K}$.
- $[a] = [b]$ if $[a] \leq [b]$ and $[b] \leq [a]$.
- $\mathcal{C}u(A) = \{[a] : a \in (A \otimes \mathcal{K})_+\}$
- $[a] + [b] := [a' + b']$ where $[a] = [a'], [b] = [b']$ and $a' \perp b'$.

The Cuntz semigroup

Ordered semigroup, constructed as follows.

- For $a, b \in (A \otimes \mathcal{K})_+$, $[a] \leq [b]$ if
$$\|a - s_n^* b s_n\| \to 0,$$
for some $(s_n) \subset A \otimes \mathcal{K}$.
- $[a] = [b]$ if $[a] \leq [b]$ and $[b] \leq [a]$.
- $\mathcal{C}u(A) = \{[a] : a \in (A \otimes \mathcal{K})_+\}$
- $[a] + [b] := [a' + b']$ where $[a] = [a'], [b] = [b']$ and $a' \perp b'$.

The Cuntz semigroup

Ordered semigroup, constructed as follows.

- For $a, b \in (A \otimes \mathcal{K})_+$, $[a] \leq [b]$ if
$$\|a - s_n^* b s_n\| \to 0,$$
for some $(s_n) \subset A \otimes \mathcal{K}$.
- $[a] = [b]$ if $[a] \leq [b]$ and $[b] \leq [a]$.
- $\mathcal{C}u(A) = \{[a] : a \in (A \otimes \mathcal{K})_+\}$
- $[a] + [b] := [a' + b']$ where $[a] = [a'], [b] = [b']$ and $a' \perp b'$.

The Cuntz semigroup

Ordered semigroup, constructed as follows.

- For $a, b \in (A \otimes \mathcal{K})_+$, $[a] \leq [b]$ if
$$\|a - s_n^* b s_n\| \to 0,$$
for some $(s_n) \subset A \otimes \mathcal{K}$.
- $[a] = [b]$ if $[a] \leq [b]$ and $[b] \leq [a]$.
- $\mathcal{C}u(A) = \{[a] : a \in (A \otimes \mathcal{K})_+\}$
- $[a] + [b] := [a' + b']$ where $[a] = [a'], [b] = [b']$ and $a' \perp b'$.

Regularity of C^*-algebras.

Theorem (Toms '08)

$\exists$ two (simple, nuclear, seperable, unital) C^*-algebras which have the same value under classical invariants, yet their Cuntz semigroups differ.

Theorem (Winter, preprint '10)

For unital, simple C^*-algebras, $\mathcal{C}u(A)$ is "nice" (almost unperforated and almost divisible) if and only if A is nice ($\mathcal{Z}$-stable and therefore, hopefully, classifiable). (Provided A has locally finite nuclear dimension).

The Cuntz semigroup: applications

Regularity of C^*-algebras.

Theorem (Toms '08)

$\exists$ two (simple, nuclear, seperable, unital) C^*-algebras which have the same value under classical invariants, yet their Cuntz semigroups differ.

Theorem (Winter, preprint '10)

For unital, simple C^*-algebras, $\mathcal{C}u(A)$ is "nice" (almost unperforated and almost divisible) if and only if A is nice ($\mathcal{Z}$-stable and therefore, hopefully, classifiable). (Provided A has locally finite nuclear dimension).

The Cuntz semigroup: applications

Regularity of C^*-algebras.

Theorem (Toms '08)

$\exists$ two (simple, nuclear, seperable, unital) C^*-algebras which have the same value under classical invariants, yet their Cuntz semigroups differ.

Theorem (Winter, preprint '10)

For unital, simple C^*-algebras, $\mathcal{C}u(A)$ is "nice" (almost unperforated and almost divisible) if and only if A is nice ($\mathcal{Z}$-stable and therefore, hopefully, classifiable). (Provided A has locally finite nuclear dimension).

The Cuntz semigroup: applications

Regularity of C^*-algebras.

Theorem (Toms '08)

$\exists$ two (simple, nuclear, seperable, unital) C^*-algebras which have the same value under classical invariants, yet their Cuntz semigroups differ.

Theorem (Winter, preprint '10)

For unital, simple C^*-algebras, $\mathcal{C}u(A)$ is "nice" (almost unperforated and almost divisible) if and only if A is nice ($\mathcal{Z}$-stable and therefore, hopefully, classifiable). (Provided A has locally finite nuclear dimension).

Classification of non-simple C^*-algebras

$\mathcal{C}u(A)$ contains the ideal lattice of A and $\mathcal{C}u(I), \mathcal{C}u(A/I)$ for every ideal I.

This makes $\mathcal{C}u(A)$ a good candidate for non-simple classification.

(Only as a part of the invariant - eg. $K_1(A)$ doesn't appear in $\mathcal{C}u(A)$.)

Theorem (Robert, preprint '10)

(Generalizing previous results by Ciuperca, Elliott, Santiago) $\mathcal{C}u(\cdot)$ classifies unital inductive limits of 1-dimensional noncommutative CW complexes with trivial K_1.

Classification of non-simple C^*-algebras

$\mathcal{C}u(A)$ contains the ideal lattice of A and $\mathcal{C}u(I), \mathcal{C}u(A/I)$ for every ideal I.

This makes $\mathcal{C}u(A)$ a good candidate for non-simple classification.

(Only as a part of the invariant - eg. $K_1(A)$ doesn't appear in $\mathcal{C}u(A)$.)

Theorem (Robert, preprint '10)

(Generalizing previous results by Ciuperca, Elliott, Santiago) $\mathcal{C}u(\cdot)$ classifies unital inductive limits of 1-dimensional noncommutative CW complexes with trivial K_1.

Classification of non-simple C^*-algebras

$\mathcal{C}u(A)$ contains the ideal lattice of A and $\mathcal{C}u(I), \mathcal{C}u(A/I)$ for every ideal I.

This makes $\mathcal{C}u(A)$ a good candidate for non-simple classification.

(Only as a part of the invariant - eg. $K_1(A)$ doesn't appear in $\mathcal{C}u(A)$.)

Theorem (Robert, preprint '10)

(Generalizing previous results by Ciuperca, Elliott, Santiago) $\mathcal{C}u(\cdot)$ classifies unital inductive limits of 1-dimensional noncommutative CW complexes with trivial K_1.

Classification of non-simple C^*-algebras

$\mathcal{C}u(A)$ contains the ideal lattice of A and $\mathcal{C}u(I), \mathcal{C}u(A/I)$ for every ideal I.

This makes $\mathcal{C}u(A)$ a good candidate for non-simple classification.

(Only as a part of the invariant - eg. $K_1(A)$ doesn't appear in $\mathcal{C}u(A)$.)

Theorem (Robert, preprint '10)

(Generalizing previous results by Ciuperca, Elliott, Santiago) $\mathcal{C}u(\cdot)$ classifies unital inductive limits of 1-dimensional noncommutative CW complexes with trivial K_1.

Classification of non-simple C^*-algebras

$\mathcal{C}u(A)$ contains the ideal lattice of A and $\mathcal{C}u(I), \mathcal{C}u(A/I)$ for every ideal I.

This makes $\mathcal{C}u(A)$ a good candidate for non-simple classification.

(Only as a part of the invariant - eg. $K_1(A)$ doesn't appear in $\mathcal{C}u(A)$.)

Theorem (Robert, preprint '10)

(Generalizing previous results by Ciuperca, Elliott, Santiago) $\mathcal{C}u(\cdot)$ classifies unital inductive limits of 1-dimensional noncommutative CW complexes with trivial K_1.

Approximately subhomogeneous algebras

A C^*-algebra is **subhomogeneous** if there is a bound on its irred. rep. dimensions.

A C^*-algebra is **approximately subhomogeneous** (ASH) if it is an inductive limit of subhomogeneous algebras.

ASH algebras include: AF algebras, irrational rotation algebras, other important (simple) C^*-algebras.

Open: Is every simple, separable, finite, nuclear C^*-algebra ASH?

Approximately subhomogeneous algebras

A C^*-algebra is **subhomogeneous** if there is a bound on its irred. rep. dimensions.

A C^*-algebra is **approximately subhomogeneous** (ASH) if it is an inductive limit of subhomogeneous algebras.

ASH algebras include: AF algebras, irrational rotation algebras, other important (simple) C^*-algebras.

Open: Is every simple, separable, finite, nuclear C^*-algebra ASH?

Approximately subhomogeneous algebras

A C^*-algebra is **subhomogeneous** if there is a bound on its irred. rep. dimensions.

A C^*-algebra is **approximately subhomogeneous** (ASH) if it is an inductive limit of subhomogeneous algebras.

ASH algebras include: AF algebras, irrational rotation algebras, other important (simple) C^*-algebras.

Open: Is every simple, separable, finite, nuclear C^*-algebra ASH?

Approximately subhomogeneous algebras

A C^*-algebra is **subhomogeneous** if there is a bound on its irred. rep. dimensions.

A C^*-algebra is **approximately subhomogeneous** (ASH) if it is an inductive limit of subhomogeneous algebras.

ASH algebras include: AF algebras, irrational rotation algebras, other important (simple) C^*-algebras.

Open: Is every simple, separable, finite, nuclear C^*-algebra ASH?

Main result: the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$

Define a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on $C_0(X, A \otimes \mathcal{K})_+$.

$\mathbb{I}(a)$ consists of:

- $x \mapsto [a(x)]$ (a function $X \mapsto \mathcal{C}u(A)$); and
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant and in $V(A)$ on K.

$\mathbb{I}(\cdot)$ is complete:

Theorem

If X is l.c., Hausdorff, 2nd countable and A is sep., ASH, $\mathcal{Z}$-stable then for $a, b \in C_0(X, A \otimes \mathcal{K})_+$,

$$[a] \leq [b] \text{ iff } \mathbb{I}(a) \leq \mathbb{I}(b)$$

(i.e. $[a(x)] \leq [b(x)]$ for all x and $[a|_K] = [b|_K]$ for any compact K for which $[a(x)] = [b(x)]$ is constant and in $V(A)$ on K).

Main result: the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$

Define a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on $C_0(X, A \otimes \mathcal{K})_+$.

$\mathbb{I}(a)$ consists of:

- $x \mapsto [a(x)]$ (a function $X \mapsto \mathcal{C}u(A)$); and
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant and in $V(A)$ on K.

$\mathbb{I}(\cdot)$ is complete:

Theorem

If X is l.c., Hausdorff, 2nd countable and A is sep., ASH, $\mathcal{Z}$-stable then for $a, b \in C_0(X, A \otimes \mathcal{K})_+$,

$$[a] \leq [b] \text{ iff } \mathbb{I}(a) \leq \mathbb{I}(b)$$

(i.e. $[a(x)] \leq [b(x)]$ for all x and $[a|_K] = [b|_K]$ for any compact K for which $[a(x)] = [b(x)]$ is constant and in $V(A)$ on K).

Main result: the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$

Define a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on $C_0(X, A \otimes \mathcal{K})_+$.

$\mathbb{I}(a)$ consists of:

- $x \mapsto [a(x)]$ (a function $X \mapsto \mathcal{C}u(A)$); and
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant and in $V(A)$ on K.

$\mathbb{I}(\cdot)$ is complete:

Theorem

If X is l.c., Hausdorff, 2nd countable and A is sep., ASH, $\mathcal{Z}$-stable then for $a, b \in C_0(X, A \otimes \mathcal{K})_+$,

$$[a] \leq [b] \text{ iff } \mathbb{I}(a) \leq \mathbb{I}(b)$$

(i.e. $[a(x)] \leq [b(x)]$ for all x and $[a|_K] = [b|_K]$ for any compact K for which $[a(x)] = [b(x)]$ is constant and in $V(A)$ on K).

Main result: the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$

Define a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on $C_0(X, A \otimes \mathcal{K})_+$.

$\mathbb{I}(a)$ consists of:

- $x \mapsto [a(x)]$ (a function $X \mapsto \mathcal{C}u(A)$); and
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant and in $V(A)$ on K.

$\mathbb{I}(\cdot)$ is complete:

Theorem

If X is l.c., Hausdorff, 2nd countable and A is sep., ASH, $\mathcal{Z}$-stable then for $a, b \in C_0(X, A \otimes \mathcal{K})_+$,

$$[a] \leq [b] \text{ iff } \mathbb{I}(a) \leq \mathbb{I}(b)$$

(i.e. $[a(x)] \leq [b(x)]$ for all x and $[a|_K] = [b|_K]$ for any compact K for which $[a(x)] = [b(x)]$ is constant and in $V(A)$ on K).

Main result: the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$

Define a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on $C_0(X, A \otimes \mathcal{K})_+$.

$\mathbb{I}(a)$ consists of:

- $x \mapsto [a(x)]$ (a function $X \mapsto \mathcal{C}u(A)$); and
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant and in $V(A)$ on K.

$\mathbb{I}(\cdot)$ is complete:

Theorem

If X is l.c., Hausdorff, 2nd countable and A is sep., ASH, $\mathcal{Z}$-stable then for $a, b \in C_0(X, A \otimes \mathcal{K})_+$,

$$[a] \leq [b] \text{ iff } \mathbb{I}(a) \leq \mathbb{I}(b)$$

(i.e. $[a(x)] \leq [b(x)]$ for all x and $[a|_K] = [b|_K]$ for any compact K for which $[a(x)] = [b(x)]$ is constant and in $V(A)$ on K).

Main result: the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$

Define a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on $C_0(X, A \otimes \mathcal{K})_+$.

$\mathbb{I}(a)$ consists of:

- $x \mapsto [a(x)]$ (a function $X \mapsto \mathcal{C}u(A)$); and
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant and in $V(A)$ on K.

$\mathbb{I}(\cdot)$ is complete:

Theorem

If X is l.c., Hausdorff, 2nd countable and A is sep., ASH, $\mathcal{Z}$-stable then for $a, b \in C_0(X, A \otimes \mathcal{K})_+$,

$$[a] \leq [b] \text{ iff } \mathbb{I}(a) \leq \mathbb{I}(b)$$

(i.e. $[a(x)] \leq [b(x)]$ for all x and $[a|_K] = [b|_K]$ for any compact K for which $[a(x)] = [b(x)]$ is constant and in $V(A)$ on K).

Main result: the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$

Define a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on $C_0(X, A \otimes \mathcal{K})_+$.

$\mathbb{I}(a)$ consists of:

- $x \mapsto [a(x)]$ (a function $X \mapsto \mathcal{C}u(A)$); and
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant and in $V(A)$ on K.

$\mathbb{I}(\cdot)$ is complete:

Theorem

If X is l.c., Hausdorff, 2nd countable and A is sep., ASH, $\mathcal{Z}$-stable then for $a, b \in C_0(X, A \otimes \mathcal{K})_+$,

$$[a] \leq [b] \text{ iff } \mathbb{I}(a) \leq \mathbb{I}(b)$$

(i.e. $[a(x)] \leq [b(x)]$ for all x and $[a|_K] = [b|_K]$ for any compact K for which $[a(x)] = [b(x)]$ is constant and in $V(A)$ on K).

Main result: the Cuntz-equivalence invariant $\mathbb{I}(\cdot)$

Define a Cuntz-equivalence invariant $\mathbb{I}(\cdot)$ on $C_0(X, A \otimes \mathcal{K})_+$.

$\mathbb{I}(a)$ consists of:

- $x \mapsto [a(x)]$ (a function $X \mapsto \mathcal{C}u(A)$); and
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant and in $V(A)$ on K.

$\mathbb{I}(\cdot)$ is complete:

Theorem

If X is l.c., Hausdorff, 2nd countable and A is sep., ASH, $\mathcal{Z}$-stable then for $a, b \in C_0(X, A \otimes \mathcal{K})_+$,

$$[a] \leq [b] \text{ iff } \mathbb{I}(a) \leq \mathbb{I}(b)$$

(i.e. $[a(x)] \leq [b(x)]$ for all x and $[a|_K] = [b|_K]$ for any compact K for which $[a(x)] = [b(x)]$ is constant and in $V(A)$ on K).

Main result: the range of $\mathbb{I}(\cdot)$

Recall: $\mathbb{I}(a)$ consists of

- $x \mapsto [a(x)] = f(x)$, a $\ll$-lower semicontinuous function $f : X \to \mathcal{C}u(A)$
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant $= [p] \in V(A)$ on K;
  for $[p] \in V(A)$, this can be captured by a single projection $a_{[p]} \in C_b(f^{-1}([p]), A \otimes \mathcal{K})$, such that $[a_{[p]}|_K] = [a|_K]$ for each compact $K \subseteq f^{-1}([p])$.
  The family $a_{[p]}$ is compatible with f in the sense that $[a_{[p]}(x)] = f(x)$ wherever defined (i.e. wherever $f(x) = [p]$).

Main result: the range of $\mathbb{I}(\cdot)$

Recall: $\mathbb{I}(a)$ consists of

- $x \mapsto [a(x)] = f(x)$, a $\ll$-lower semicontinuous function $f : X \to \mathcal{C}u(A)$
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant $= [p] \in V(A)$ on K;

  for $[p] \in V(A)$, this can be captured by a single projection $a_{[p]} \in C_b(f^{-1}([p]), A \otimes \mathcal{K})$, such that $[a_{[p]}|_K] = [a|_K]$ for each compact $K \subseteq f^{-1}([p])$.
  The family $a_{[p]}$ is compatible with f in the sense that $[a_{[p]}(x)] = f(x)$ wherever defined (i.e. wherever $f(x) = [p]$).

Recall: $\mathbb{I}(a)$ consists of

- $x \mapsto [a(x)] = f(x)$, a $\ll$-lower semicontinuous function $f : X \to \mathcal{C}u(A)$
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant $= [p] \in V(A)$ on K;
  for $[p] \in V(A)$, this can be captured by a single projection $a_{[p]} \in C_b(f^{-1}([p]), A \otimes \mathcal{K})$, such that $[a_{[p]}|_K] = [a|_K]$ for each compact $K \subseteq f^{-1}([p])$.
  The family $a_{[p]}$ is compatible with f in the sense that $[a_{[p]}(x)] = f(x)$ wherever defined (i.e. wherever $f(x) = [p]$).

Main result: the range of $\mathbb{I}(\cdot)$

Recall: $\mathbb{I}(a)$ consists of

- $x \mapsto [a(x)] = f(x)$, a $\ll$-lower semicontinuous function $f : X \to \mathcal{C}u(A)$
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant $= [p] \in V(A)$ on K;
  for $[p] \in V(A)$, this can be captured by a single projection $a_{[p]} \in C_b(f^{-1}([p]), A \otimes \mathcal{K})$, such that $[a_{[p]}|_K] = [a|_K]$ for each compact $K \subseteq f^{-1}([p])$.
  The family $a_{[p]}$ is compatible with f in the sense that $[a_{[p]}(x)] = f(x)$ wherever defined (i.e. wherever $f(x) = [p]$).

Main result: the range of $\mathbb{I}(\cdot)$

Recall: $\mathbb{I}(a)$ consists of

- $x \mapsto [a(x)] = f(x)$, a $\ll$-lower semicontinuous function $f : X \to \mathcal{C}u(A)$
- $[a|_K] \in V(C(K, A))$ for each compact $K \subset X$ for which $[a(x)]$ is constant $= [p] \in V(A)$ on K;
  for $[p] \in V(A)$, this can be captured by a single projection $a_{[p]} \in C_b(f^{-1}([p]), A \otimes \mathcal{K})$, such that $[a_{[p]}|_K] = [a|_K]$ for each compact $K \subseteq f^{-1}([p])$.
  The family $a_{[p]}$ is compatible with f in the sense that $[a_{[p]}(x)] = f(x)$ wherever defined (i.e. wherever $f(x) = [p]$).

Main result: the range of $\mathbb{I}(\cdot)$

Theorem

X, A as before. Given a $\ll$-lower semicontinuous $f : X \to \mathcal{C}u(A)$ and a compatible family of projections $(a_{[p]})_{[p]\in V(A)}$, there exists $[a] \in \mathcal{C}u(C_0(X, A))$ such that $\mathbb{I}(a) = (f, (a_{[p]})_{[p]\in V(A)})$; that is,

- $[a(x)] = f(x)$ for all x; and
- $[a|_K] = [a_{[p]}|_K]$ for any compact $K \subseteq f^{-1}([p])$.

Main result: the range of $\mathbb{I}(\cdot)$

Theorem

X, A as before. Given a $\ll$-lower semicontinuous $f : X \to \mathcal{C}u(A)$ and a compatible family of projections $(a_{[p]})_{[p]\in V(A)}$, there exists $[a] \in \mathcal{C}u(C_0(X, A))$ such that $\mathbb{I}(a) = (f, (a_{[p]})_{[p]\in V(A)})$; that is,

- $[a(x)] = f(x)$ for all x; and
- $[a|_K] = [a_{[p]}|_K]$ for any compact $K \subseteq f^{-1}([p])$.

Main result: the range of $\mathbb{I}(\cdot)$

Theorem

X, A as before. Given a $\ll$-lower semicontinuous $f : X \to \mathcal{C}u(A)$ and a compatible family of projections $(a_{[p]})_{[p] \in V(A)}$, there exists $[a] \in \mathcal{C}u(C_0(X, A))$ such that $\mathbb{I}(a) = (f, (a_{[p]})_{[p] \in V(A)}$; that is,

- $[a(x)] = f(x)$ for all x; and
- $[a|_K] = [a_{[p]}|_K]$ for any compact $K \subseteq f^{-1}([p])$.

Main result: the range of $\mathbb{I}(\cdot)$

Theorem

X, A as before. Given a $\ll$-lower semicontinuous $f : X \to \mathcal{C}u(A)$ and a compatible family of projections $(a_{[p]})_{[p]\in V(A)}$, there exists $[a] \in \mathcal{C}u(C_0(X, A))$ such that $\mathbb{I}(a) = (f, (a_{[p]})_{[p]\in V(A)}$; that is,

- $[a(x)] = f(x)$ for all x; and
- $[a|_K] = [a_{[p]}|_K]$ for any compact $K \subseteq f^{-1}([p])$.

Main result: putting it together

Theorem

A, X as before. The Cuntz semigroup of $C_0(X, A)$ may be identified with pairs $(f, (\langle a_{[p]} \rangle)_{[p] \in V(A)})$ where

- $f : X \to \mathcal{C}u(A)$ is $\ll$-lower semicontinuous; and
- For each $[p] \in V(A)$, $a_{[p]}$ is a projection in $C_b(f^{-1}([p]), A \otimes \mathcal{K})$ satisfying $[a_{[p]}(x)] = [p]$ for all x, and $\langle a_{[p]} \rangle$ denotes its equivalence class via the relation $a_{[p]} \sim b_{[p]}$ if $a_{[p]}|_K \sim_{M-vN} b_{[p]}|_K$ for all compact $K \subseteq f^{-1}([p])$.

Main result: putting it together

Theorem

A, X as before. The Cuntz semigroup of $C_0(X, A)$ may be identified with pairs $(f, (\langle a_{[p]}\rangle)_{[p]\in V(A)})$ where

- $f : X \to \mathcal{C}u(A)$ is $\ll$-lower semicontinuous; and
- For each $[p] \in V(A)$, $a_{[p]}$ is a projection in $C_b(f^{-1}([p]), A \otimes \mathcal{K})$ satisfying $[a_{[p]}(x)] = [p]$ for all x, and $\langle a_{[p]}\rangle$ denotes its equivalence class via the relation $a_{[p]} \sim b_{[p]}$ if $a_{[p]}|_K \sim_{M-vN} b_{[p]}|_K$ for all compact $K \subseteq f^{-1}([p])$.

Main result: putting it together

Theorem

A, X as before. The Cuntz semigroup of $C_0(X, A)$ may be identified with pairs $(f, (\langle a_{[p]} \rangle)_{[p] \in V(A)})$ where

- $f : X \to \mathcal{C}u(A)$ is $\ll$-lower semicontinuous; and
- For each $[p] \in V(A)$, $a_{[p]}$ is a projection in $C_b(f^{-1}([p]), A \otimes \mathcal{K})$ satisfying $[a_{[p]}(x)] = [p]$ for all x, and $\langle a_{[p]} \rangle$ denotes its equivalence class via the relation $a_{[p]} \sim b_{[p]}$ if $a_{[p]}|_K \sim_{M-vN} b_{[p]}|_K$ for all compact $K \subseteq f^{-1}([p])$.

Main result: putting it together

Theorem

A, X as before. The Cuntz semigroup of $C_0(X, A)$ may be identified with pairs $(f, (\langle a_{[p]}\rangle)_{[p]\in V(A)})$ where

- $f : X \to \mathcal{C}u(A)$ is $\ll$-lower semicontinuous; and
- For each $[p] \in V(A)$, $a_{[p]}$ is a projection in $C_b(f^{-1}([p]), A \otimes \mathcal{K})$ satisfying $[a_{[p]}(x)] = [p]$ for all x, and $\langle a_{[p]}\rangle$ denotes its equivalence class via the relation $a_{[p]} \sim b_{[p]}$ if $a_{[p]}|_K \sim_{M-vN} b_{[p]}|_K$ for all compact $K \subseteq f^{-1}([p])$.

Structure of ASH algebras

Proposition (Ng-Winter, Phillips, T)

Every separable ASH algebra is an inductive limit of RSH algebras with finite dimensional total space.

My contribution: extending the result to the nonunital case.

Structure of ASH algebras

Proposition (Ng-Winter, Phillips, T)

Every separable ASH algebra is an inductive limit of RSH algebras with finite dimensional total space.

My contribution: extending the result to the nonunital case.

Structure of ASH algebras

Proposition (Ng-Winter, Phillips, T)

Every separable ASH algebra is an inductive limit of RSH algebras with finite dimensional total space.

My contribution: extending the result to the nonunital case.

Structure of ASH algebras

Proposition

Let A be RSH with finite dimensional total space, $I \subseteq A$ an ideal. Suppose $a, b \in A$ are such that

- $\mathrm{Rank}\,\sigma(a) < \mathrm{Rank}\,\sigma(b)$ for every irred. rep. $\sigma : A \to M_n$ for which $\sigma(I) = 0$; and
- $[\pi_I(a)] \leq [\pi_I(b)]$ in $\mathcal{C}u(A/I)$.

Then $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(A \otimes \mathcal{Z})$.

This result is used on finite stages to achieve the completeness of $\mathbb{I}(\cdot)$.
A slightly stronger version is also used in the computation of the range of $\mathbb{I}(\cdot)$.

Structure of ASH algebras

Proposition

Let A be RSH with finite dimensional total space, $I \subseteq A$ an ideal. Suppose $a, b \in A$ are such that

- $\mathrm{Rank}\,\sigma(a) < \mathrm{Rank}\,\sigma(b)$ for every irred. rep. $\sigma : A \to M_n$ for which $\sigma(I) = 0$; and
- $[\pi_I(a)] \leq [\pi_I(b)]$ in $\mathcal{C}u(A/I)$.

Then $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(A \otimes \mathcal{Z})$.

This result is used on finite stages to achieve the completeness of $\mathbb{I}(\cdot)$.

A slightly stronger version is also used in the computation of the range of $\mathbb{I}(\cdot)$.

Structure of ASH algebras

Proposition

Let A be RSH with finite dimensional total space, $I \subseteq A$ an ideal. Suppose $a, b \in A$ are such that

- $\mathrm{Rank}\,\sigma(a) < \mathrm{Rank}\,\sigma(b)$ for every irred. rep. $\sigma : A \to M_n$ for which $\sigma(I) = 0$; and
- $[\pi_I(a)] \leq [\pi_I(b)]$ in $\mathcal{C}u(A/I)$.

Then $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(A \otimes \mathcal{Z})$.

This result is used on finite stages to achieve the completeness of $\mathbb{I}(\cdot)$.

A slightly stronger version is also used in the computation of the range of $\mathbb{I}(\cdot)$.

Structure of ASH algebras

Proposition

Let A be RSH with finite dimensional total space, $I \subseteq A$ an ideal. Suppose $a, b \in A$ are such that

- $\mathrm{Rank}\,\sigma(a) < \mathrm{Rank}\,\sigma(b)$ for every irred. rep. $\sigma : A \to M_n$ for which $\sigma(I) = 0$; and
- $[\pi_I(a)] \leq [\pi_I(b)]$ in $\mathcal{C}u(A/I)$.

Then $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(A \otimes \mathcal{Z})$.

This result is used on finite stages to achieve the completeness of $\mathbb{I}(\cdot)$.

A slightly stronger version is also used in the computation of the range of $\mathbb{I}(\cdot)$.

Structure of ASH algebras

Proposition

Let A be RSH with finite dimensional total space, $I \subseteq A$ an ideal. Suppose $a, b \in A$ are such that

- $\mathrm{Rank}\,\sigma(a) < \mathrm{Rank}\,\sigma(b)$ for every irred. rep. $\sigma : A \to M_n$ for which $\sigma(I) = 0$; and
- $[\pi_I(a)] \leq [\pi_I(b)]$ in $\mathcal{C}u(A/I)$.

Then $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(A \otimes \mathcal{Z})$.

This result is used on finite stages to achieve the completeness of $\mathbb{I}(\cdot)$.

A slightly stronger version is also used in the computation of the range of $\mathbb{I}(\cdot)$.

Structure of ASH algebras

Proposition

Let A be RSH with finite dimensional total space, $I \subseteq A$ an ideal. Suppose $a, b \in A$ are such that

- $\mathrm{Rank}\,\sigma(a) < \mathrm{Rank}\,\sigma(b)$ for every irred. rep. $\sigma : A \to M_n$ for which $\sigma(I) = 0$; and
- $[\pi_I(a)] \leq [\pi_I(b)]$ in $\mathcal{C}u(A/I)$.

Then $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(A \otimes \mathcal{Z})$.

This result is used on finite stages to achieve the completeness of $\mathbb{I}(\cdot)$.

A slightly stronger version is also used in the computation of the range of $\mathbb{I}(\cdot)$.

Structure of ASH algebras

Proposition

Let A be RSH with finite dimensional total space, $I \subseteq A$ an ideal. Suppose $a, b \in A$ are such that

- $\mathrm{Rank}\,\sigma(a) < \mathrm{Rank}\,\sigma(b)$ for every irred. rep. $\sigma : A \to M_n$ for which $\sigma(I) = 0$; and
- $[\pi_I(a)] \leq [\pi_I(b)]$ in $\mathcal{C}u(A/I)$.

Then $[a \otimes 1_{\mathcal{Z}}] \leq [b \otimes 1_{\mathcal{Z}}]$ in $\mathcal{C}u(A \otimes \mathcal{Z})$.

This result is used on finite stages to achieve the completeness of $\mathbb{I}(\cdot)$.
A slightly stronger version is also used in the computation of the range of $\mathbb{I}(\cdot)$.

Riesz interpolation in $\mathcal{C}u(A)$

Proposition (Brown-Perera-Toms, Elliott-Robert-Santiago)

If A is simple, finite, exact, and $\mathcal{Z}$-stable then

$$\mathcal{C}u(A) \cong V(A) \amalg Lsc(T(A), (0, \infty]).$$

Proposition

For A as above, $\mathcal{C}u(A)$ has Riesz interpolation , i.e. if

$$\begin{matrix} [a_1] \\ [a_2] \end{matrix} \leq \begin{matrix} [c_1] \\ [c_2], \end{matrix}$$

then there exists $[b] \in \mathcal{C}u(A)$ such that

$$\begin{matrix} [a_1] \\ [a_2] \end{matrix} \leq [b] \leq \begin{matrix} [c_1] \\ [c_2]. \end{matrix}$$

Riesz interpolation in $\mathcal{C}u(A)$

Proposition (Brown-Perera-Toms, Elliott-Robert-Santiago)

If A is simple, finite, exact, and $\mathcal{Z}$-stable then

$$\mathcal{C}u(A) \cong V(A) \amalg Lsc(T(A), (0, \infty]).$$

Proposition

For A as above, $\mathcal{C}u(A)$ has Riesz interpolation , i.e. if

$$\begin{matrix} [a_1] \\ [a_2] \end{matrix} \leq \begin{matrix} [c_1] \\ [c_2], \end{matrix}$$

then there exists $[b] \in \mathcal{C}u(A)$ such that

$$\begin{matrix} [a_1] \\ [a_2] \end{matrix} \leq [b] \leq \begin{matrix} [c_1] \\ [c_2]. \end{matrix}$$

Riesz interpolation in $\mathcal{C}u(A)$

Proposition (Brown-Perera-Toms, Elliott-Robert-Santiago)

If A is simple, finite, exact, and $\mathcal{Z}$-stable then
$$\mathcal{C}u(A) \cong V(A) \amalg Lsc(T(A), (0, \infty]).$$

Proposition

For A as above, $\mathcal{C}u(A)$ has Riesz interpolation , i.e. if
$$\begin{matrix}[a_1] \\ [a_2]\end{matrix} \leq \begin{matrix}[c_1] \\ [c_2],\end{matrix}$$
then there exists $[b] \in \mathcal{C}u(A)$ such that
$$\begin{matrix}[a_1] \\ [a_2]\end{matrix} \leq [b] \leq \begin{matrix}[c_1] \\ [c_2].\end{matrix}$$

Riesz interpolation in $\mathcal{C}u(A)$

Proposition (Brown-Perera-Toms, Elliott-Robert-Santiago)

If A is simple, finite, exact, and $\mathcal{Z}$-stable then
$$\mathcal{C}u(A) \cong V(A) \amalg Lsc(T(A), (0, \infty]).$$

Proposition

For A as above, $\mathcal{C}u(A)$ has Riesz interpolation , i.e. if
$$\begin{matrix}[a_1] \\ [a_2]\end{matrix} \leq \begin{matrix}[c_1] \\ [c_2]\end{matrix},$$
then there exists $[b] \in \mathcal{C}u(A)$ such that
$$\begin{matrix}[a_1] \\ [a_2]\end{matrix} \leq [b] \leq \begin{matrix}[c_1] \\ [c_2]\end{matrix}.$$

Riesz interpolation in $\mathcal{C}u(A)$

Proposition (Brown-Perera-Toms, Elliott-Robert-Santiago)

If A is simple, finite, exact, and $\mathcal{Z}$-stable then

$$\mathcal{C}u(A) \cong V(A) \amalg Lsc(T(A), (0, \infty]).$$

Proposition

For A as above, $\mathcal{C}u(A)$ has Riesz interpolation , i.e. if

$$\begin{matrix}[a_1] \\ [a_2]\end{matrix} \le \begin{matrix}[c_1] \\ [c_2]\end{matrix},$$

then there exists $[b] \in \mathcal{C}u(A)$ such that

$$\begin{matrix}[a_1] \\ [a_2]\end{matrix} \le [b] \le \begin{matrix}[c_1] \\ [c_2]\end{matrix}.$$

Riesz interpolation in $\mathcal{C}u(A)$

Proposition

For A as above, $\mathcal{C}u(A)$ has Riesz interpolation

This result is used in the computation of the range of $\mathbb{I}(\cdot)$:

Lemma

If A is separable, $\mathcal{C}u(A)$ has Riesz interpolation, and $f : X \to \mathcal{C}u(A)$ is $\ll$-lower semicontinuous then f is the pointwise supremum of a sequence of $\ll$-lower semicontinuous functions $f_n : X \to \mathcal{C}u(A)$ with finite range.

Riesz interpolation in $\mathcal{C}u(A)$

Proposition

For A as above, $\mathcal{C}u(A)$ has Riesz interpolation

This result is used in the computation of the range of $\mathbb{I}(\cdot)$:

Lemma

If A is separable, $\mathcal{C}u(A)$ has Riesz interpolation, and $f : X \to \mathcal{C}u(A)$ is $\ll$-lower semicontinuous then f is the pointwise supremum of a sequence of $\ll$-lower semicontinuous functions $f_n : X \to \mathcal{C}u(A)$ with finite range.

Riesz interpolation in $\mathcal{C}u(A)$

Proposition

For A as above, $\mathcal{C}u(A)$ has Riesz interpolation

This result is used in the computation of the range of $\mathbb{I}(\cdot)$:

Lemma

If A is separable, $\mathcal{C}u(A)$ has Riesz interpolation, and $f : X \to \mathcal{C}u(A)$ is $\ll$-lower semicontinuous then f is the pointwise supremum of a sequence of $\ll$-lower semicontinuous functions $f_n : X \to \mathcal{C}u(A)$ with finite range.