NUCLEAR DIMENSION AND $\mathcal{Z}$-STABILITY OF NON-SIMPLE C*-ALGEBRAS

LEONEL ROBERT AND AARON TIKUISIS

Abstract. We investigate the conjecture that finite nuclear dimension implies $\mathcal{Z}$-stability, for sufficiently non-type I, separable C*-algebras. We prove this conjecture in the following cases: (i) the C*-algebra has no purely infinite subquotients and its primitive ideal space has a basis of compact open sets, (ii) the C*-algebra has no purely infinite quotients and its primitive ideal space is Hausdorff. Along the way, we show that, in the presence of appropriate finiteness assumptions, finite nuclear dimension implies algebraic regularity properties in the Cuntz semigroup. Furthermore, these algebraic regularity properties, together with locally finite nuclear dimension and (i) or (ii), imply $\mathcal{Z}$-stability. A crucial tool we develop is a certain factorization of the identity map on the central sequence algebra, in close analogy with the definition of nuclear dimension.

1. Introduction

Many recent advances in the study and classification of nuclear C*-algebras have centered around understanding low-dimensional behaviour or regularity. Examples of Rørdam [20] and Toms [30], relying on techniques pioneered by Villadsen [32], demonstrated that some sort of regularity condition, stronger than nuclearity, is necessary in order to have a classification by K-theory and traces. Three candidate regularity conditions, involving quite diverse ideas, have been introduced: finite nuclear dimension, tensorial absorption of the Jiang-Su algebra $\mathcal{Z}$, and regularity in the Cuntz semigroup; see [5] for an overview. A great deal of effort in current C*-algebras research has gone into showing that these properties are equivalent as broadly as possible – i.e., avoiding natural obstructions, such as being type I, non-nuclear, or non-separable – and there has been significant progress towards this goal [10, 12, 13, 24, 27–29, 35, 37]. This article contributes further to this goal by developing systematic tools to prove $\mathcal{Z}$-stability from other regularity conditions.

In the case of simple, unital (and as always, separable and non-type I) C*-algebras, Winter showed in [37] finite nuclear dimension implies $\mathcal{Z}$-stability. Furthermore, the same conclusion holds if the assumption of finite nuclear dimension is replaced by having locally finite nuclear dimension and being (M, N)-pure (meaning that the algebra's Cuntz semigroup has the regularity properties of M-comparison and N-almost-divisibility). In [27], the second-named author extended these results to the non-unital, simple case. Firstly, this article clarifies these arguments, emphasizing the role of Cuntz-semigroup-regularity in the central sequence algebra.

Can these results be extended to non-simple C*-algebras? This question is addressed by the following conjectures, which have been the main motivation in our investigations:

(C1) If a separable C*-algebra has finite nuclear dimension and no elementary subquotients then it is $\mathcal{Z}$-stable.

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(C2) If a separable C*-algebra has locally finite nuclear dimension and is (M,N)-pure for some $M,N>0$ then it is $\mathcal{Z}$-stable.

In (C1) and throughout, we use "subquotient" to mean the quotient of an ideal (rather than the quotient of a subalgebra). Notice that the first conjecture follows from the second, provided that one shows that C*-algebras of finite nuclear dimension and without elementary subquotients are (M,N)-pure for some $M,N>0$; this is the path followed in [37], [27], and also here. More precisely, we show in Theorem 3.1 that if a C*-algebra of finite nuclear dimension has no elementary subquotients *and no purely infinite subquotients* then it is (M,N)-pure for some $M,N>0$. We also show that such a C*-algebra has strong tracial M-comparison, even without the hypothesis of having no elementary subquotients (Theorem 3.7). Then, in Theorems 7.9 and 7.14, Conjectures (C1) and (C2) above are proven under the following additional assumptions:

(A1) no simple subquotient of the C*-algebra is purely infinite, and
(A2) the primitive spectrum of the C*-algebra satisfies either one of the following
  (a) it has a basis of compact open sets, or
  (b) it is Hausdorff.

Every C*-algebra of locally finite decomposition rank and with the ideal property (i.e., such that every closed two-sided ideal is generated by its projections) satisfies (A1) and (A2)(a). Crossed products of the Cantor set by a free $\mathbb{Z}^n$ action also satisfy (A1) and (A2)(a), and have been shown by Szabó to have finite nuclear dimension [26]. The case (A2)(b) complements the main result of [28], and can be used to understand the range of possibilities of $C(X)$-algebras with strongly self-absorbing fibres, such as the examples in [7]. A simpler proof of $\mathcal{Z}$-stability in the simple finite case, is presented in Section 7.1. This section also contains a separate argument for the simple purely infinite case, that does not appeal to Kirchberg's $\mathcal{O}_\infty$-absorption theorem.

A main tool in proving these results is a factorization of the identity map on $(A_\omega \cap B')/B^\perp$ by c.p.c. order zero maps, when $B \subset A_\omega$ is a separable C*-subalgebra of finite nuclear dimension (but with no assumptions on A) (Theorem 4.1). This allows us to turn comparison and divisibility statements about A (or A_ω) into weaker ones about $(A_\omega \cap B')/B^\perp$ (as is done, for example, in Proposition 5.4 and Theorem 6.1 respectively).

The obstacle to the complete resolution of the conjectures above is the construction of full orthogonal elements in the central sequence algebra. Certainly, if A is $\mathcal{Z}$-stable then one can easily see that $(A_\omega \cap A')/A^\perp$ contains orthogonal full elements. On the other hand, Theorem 7.8 implies that if A is separable, has finite nuclear dimension and its central sequence algebra has two orthogonal full elements, then it is $\mathcal{Z}$-stable.

Our approach to constructing full orthogonal elements makes use of the finiteness conditions (A1) and (A2), which ensure that certain orthogonal elements obtained using Kirchberg's covering number and functional calculus are indeed full (see Lemmas 3.4, 6.4, and 7.13). Example 3.5 shows definitively that this construction cannot work without some kind of finiteness condition. The same construction was used by Winter in [37], so that finiteness (or the existence of a nontrivial trace) also underpins the arguments there.

Even the following (much) weaker question remains open:

Question 1.1. If A is of finite nuclear dimension and without elementary quotients, does A contain two (almost) full orthogonal elements?

What is meant by A having two *almost* full orthogonal elements is that, given any element a of the Pedersen ideal of A, there exist two orthogonal elements, both of which generate an ideal containing a. (It is equivalent to having two full orthogonal elements when $\mathrm{Prim}(A)$ is compact.)

This paper is organized as follows: In Section 2 we cover, among other preliminary facts, algebraic regularity properties of the Cuntz semigroup, the notion of nuclear dimension, and a criterion for $\mathcal{Z}$-stability involving the central sequence algebra. In Section 3 we investigate the divisibility properties of C*-algebras of finite nuclear dimension. We apply these results to give a simple proof of Dadarlat and Toms's result on the $\mathcal{Z}$-stability of infinite tensor products [4]. In Section 4 we prove the above mentioned factorization of the identity on central sequence algebras. In Sections 5 and 6 we apply this factorization to investigate comparison and divisibility properties of central sequence algebras. Finally, Section 7 contains the proofs of $\mathcal{Z}$-stability.

2. Preliminaries

Let us start by fixing some of the notation that will be used throughout the paper. Let A be a C*-algebra. We denote by A_+ the cone of positive elements of A and by $A^\sim$ the unitization of A. Let $a \in A_+$. The hereditary subalgebra $\overline{aAa}$ will be denoted by $\mathrm{her}(a)$. If $\varepsilon > 0$ then $(a-\varepsilon)_+$ denotes the element obtained by functional calculus evaluating the function $(t-\varepsilon)_+ := \max(t-\varepsilon, 0)$, with $t \geqslant 0$, on the positive element a. We will also frequently use functional calculus with the function $g_\varepsilon \in C_0(0,1]$ which is 0 on $[0, \frac{\varepsilon}{2}]$, 1 on $[\varepsilon, 1]$ and linear otherwise.

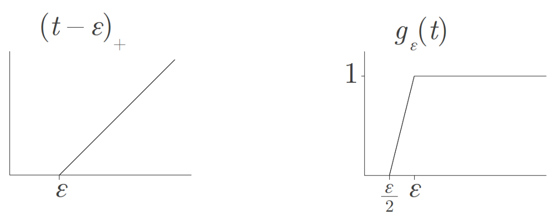

Figure 1. Graphs of $(t-\varepsilon)_+$ and $g_\varepsilon(t)$

Let $a, b \in A$. Let us write $a \approx_\varepsilon b$ to indicate that $\|a-b\| < \varepsilon$. The commutator $ab - ba$ is denoted by $[a,b]$. If $\alpha\colon A \to B$ is a linear map between C*-algebras and $b \in B$ then $\|[\alpha, b]\| < \varepsilon$ means that $\|[\alpha(v), b]\| < \varepsilon$ for all contractions $v \in A$. If $\beta\colon A \to B$ is another map then $\|[\alpha,\beta]\| < \varepsilon$ means that $\|[\alpha(v), \beta(w)]\| < \varepsilon$ for all contractions $v, w \in A$.

A linear map $\tau\colon A_+ \to [0,\infty]$ is called a trace on A if $\tau(0) = 0$ and $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. The cone of lower semicontinuous traces on A is denoted by $\mathrm{T}(A)$. We emphasize that $\mathrm{T}(A)$ does not denote only the set of bounded traces (or even tracial states), even though that convention has often been used in the literature. Lower semicontinuous traces on A extended uniquely to lower semicontinuous traces on $A \otimes \mathcal{K}$. Thus, we will assume tacitly that the domain of the traces in $\mathrm{T}(A)$ is $(A \otimes \mathcal{K})_+$. Here, and throughout the paper, $\mathcal{K}$ denotes the C*-algebra of compact operators on a separable Hilbert space.

2.1. **The Cuntz semigroup.** We will make frequent use of the arithmetic of Cuntz classes of positive elements. Let us recall the definition of the Cuntz semigroup. Let A be a C*-algebra. Let $a, b \in A_+$. Then a is said to be Cuntz smaller than b, denoted by $a \precsim b$, if there exist $d_n \in A$ such that $d_n^* b d_n \to a$; a and b are Cuntz equivalent, denoted by $a \sim b$, if $a \precsim b$ and $b \precsim a$. The relation $\precsim$ is a pre-order relation and, consequently, $\sim$ is an equivalence relation.

The Cuntz semigroup of the C*-algebra A is defined as the set of Cuntz equivalence classes of positive element of $A\otimes\mathcal{K}$. If $a\in(A\otimes\mathcal{K})_+$, the Cuntz class of a is denoted by $[a]$. The relation $[a]\leqslant[b]$ if $a\precsim[b]$ defines an order on $\mathrm{Cu}(A)$. The addition operation on $\mathrm{Cu}(A)$ is such that $[a]+[b]=[a'+b']$, where $a\sim a'$, $b\sim b'$, and $a'b'=0$ (such elements can always be found using the stability of $A\otimes\mathcal{K}$).

A positive element $a\in(A\otimes\mathcal{K})_+$, and its Cuntz class $[a]$, are called **properly infinite** if $a\neq 0$ and $2[a]\leqslant[a]$ in $\mathrm{Cu}(A)$. They are called **stably properly infinite** if $a\neq 0$ and for some $n\in\mathbb{N}$, $(n+1)[a]\leqslant n[a]$ (equivalently, for some $n\in\mathbb{N}$, $n[a]$ is properly infinite).

Let $\tau\in\mathrm{T}(A)$ be a lower semicontinuous trace on A. For each $a\in(A\otimes\mathcal{K})_+$ let us define

$$d_\tau(a)=\lim_n\tau(a^{\frac{1}{n}}).$$

The number $d_\tau(a)$ depends only on the Cuntz class of a and is understood as giving rise to an additive, order preserving, and supremum preserving map on $\mathrm{Cu}(A)$ (a.k.a., a functional on $\mathrm{Cu}(A)$) given by $[a]\mapsto d_\tau(a)$. This holds more generally when τ is a lower semicontinuous 2-quasitrace on A (see [1, Section II]). A theorem of Haagerup says that if A is an exact C*-algebra (in particular, if it is nuclear), then a lower semicontinuous 2-quasitrace on A is a trace. However, we will often use 2-quasitraces instead of traces in order to state our results in more generality. The cone of lower semicontinuous 2-quasitraces on A will be denoted by $\mathrm{QT}(A)$, and when we will simply say "quasitrace" to mean lower semicontinuous 2-quasitrace.

Let $[a],[b]\in\mathrm{Cu}(A)$ and $\gamma>0$. We write $[a]\propto[b]$ to mean that $[a]\leqslant n[b]$ for some $n\in\mathbb{N}$. We write $[a]<_s\gamma[b]$ to mean that there exists $\gamma'<\gamma$ such that $d_\tau(a)\leqslant\gamma' d_\tau(b)$ for all $\tau\in\mathrm{QT}(A)$. In the case $\gamma=1$, the relation $<_s$ has been defined elsewhere in the literature with a slightly different meaning; see for example [15, Definition 2.2].

2.2. **The central sequence algebra.** Let $(A_k)_{k=1}^\infty$ be a sequence of C*-algebras. Let us denote by $\prod_{k=1}^\infty A_k$ the C*-algebra of norm-bounded sequences $(a_k)_{k=1}^\infty$ with $a_k\in A_k$ for all k. Let ω be a free ultrafilter in $\mathbb{N}$. Let us denote by $c_\omega((A_k)_{k=1}^\infty)$ the closed two-sided ideal of $\prod_{k=1}^\infty A_k$ of sequences $(a_k)_{k=1}^\infty$ for which $\lim_\omega\|a_k\|=0$. The **ultraproduct** of the C*-algebras A_k, $k=1,2,\dots$, is defined as

$$\prod_\omega A_k:=\Big(\prod_{k=1}^\infty A_k\Big)/c_\omega((A_k)_{k=1}^\infty).$$

Whenever it is clear by the context, we will denote the quotient map from $\prod_{k=1}^\infty A_k$ to $\prod_\omega A_k$ by π_ω. If $A_k=A$ for all $k=1,\dots$ we denote the ultraproduct by A_ω and call it the **ultrapower** of A.

Observe that A embeds inside A_ω as the set of constant sequences. Let us denote by $A'\cap A_\omega$ the commutant of A inside A_ω, i.e., the elements of $a\in A_\omega$ such that $[a,c]=0$ for all $c\in A$. Let us denote by $A^\perp\cap A_\omega$ (or sometimes simply $A^\perp$) the elements of A_ω that are orthogonal to A, i.e., the elements $a\in A_\omega$ such that $ac=ca=0$ for all $c\in A$. Observe that $A^\perp$ is a a closed two-sided ideal of $A'\cap A_\omega$. The central sequence C*-algebra is defined as

$$\mathrm{F}(A):=(A'\cap A_\omega)/A^\perp.$$

We will also consider the following more general central sequence algebras (studied by Kirchberg in [9]): let $B\subseteq A_\omega$ be a C*-subalgebra. Let us denote by $B'\cap A_\omega$ its commutant and by $B^\perp$ the subalgebra of A_ω of elements orthogonal to B. The algebra $B^\perp$ is again an ideal of $B'\cap A_\omega$. We define

$$\mathrm{F}(B,A):=(B'\cap A_\omega)/B^\perp.$$

2.3. **Divisibility and comparison.** Algebraic regularity properties – of comparison and divisibility – in the Cuntz semigroup of a C*-algebra play a key role in our arguments. Here we recall M-comparison and N-almost divisibility, which together form the notion of (M,N)-pureness.

Let $M \in \mathbb{N}$. Let us say that A has M**-comparison** if for all $[a],[b_0],[b_1],\dots,[b_M] \in \mathrm{Cu}(A)$ we have that $[a] <_s [b_i]$ for $i = 0,\dots,m$ implies that $[a] \leqslant \sum_{i=0}^M [b_i]$.

Let $N \in \mathbb{N}$. Let us say that A is N**-almost divisible** if for each $[a] \in \mathrm{Cu}(A)$, $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $[b] \in \mathrm{Cu}(A)$ such that

$$k \cdot [b] \leqslant [a] \text{ and } [(b-\varepsilon)_+] \leqslant (k+1)(N+1)[b].$$

Following Winter [37], we call a C*-algebra (M,N)**-pure** if it has M-comparison and is N-almost divisible. (We point out, however, that our definition of N-almost divisibility does not exactly agree with Winter's.) The comparison and divisibility properties on $\mathrm{Cu}(A)$ relate to $\mathcal{Z}$-stability and nuclear dimension: If A has nuclear dimension m, then it has m-comparison [17], while if A is $\mathcal{Z}$-stable then it is $(0,0)$-pure [37, Proposition 3.7] (cf. also Conjecture (C2) and the remarks following it, above).

Lemma 2.1. *The following are equivalent.*

(i) A is N-almost divisible.

(ii) For every $e, a \in A_+, k \in \mathbb{N}$ and $\varepsilon > 0$, there exist $v \in M_{(k+1)(N+1)\times 1}(A)$ and a c.p.c. order zero map $\phi\colon M_k(\mathbb{C}) \to A$ such that $e\phi(\cdot) = \phi$, $(a-\varepsilon)_+ = v^*v$ and $v = (\phi(e_{11}) \otimes 1_{(k+1)(N+1)})v$.

Proof. (i) $\Rightarrow$ (ii): Since A is N-almost divisible, there exists $b \in A_+$ such that $k[b] \leqslant [a]$ and $[(a-\frac{\varepsilon}{2})_+] \leqslant (k+1)(N+1)[b]$. Let $\delta > 0$ and $x \in M_{(k+1)(N+1)\times 1}(A)$ be such that $(a-\varepsilon)_+ = x^*((b-\delta)_+ \otimes 1_{(k+1)(N+1)})x$. By [18, Lemma 2.4] and [19, Proposition 2.4], there exist a c.p.c. order zero map $\tilde{\phi}\colon M_k(\mathbb{C}) \to \mathrm{her}(a)$, $\eta > 0$, and $y \in A$ such that $(b-\delta)_+ = y^*(\tilde{\phi}(e_{11}) - \eta)_+ y$. Setting $\phi := g_\eta(\tilde{\phi})$ and $v := ((\tilde{\phi}(e_{11}) - \eta)_+^{1/2} y \otimes 1_{(k+1)(N+1)})x$ we easily see that the properties in (ii) hold.

(ii) $\Rightarrow$ (i): Let us apply (ii) to $(a-\frac{\varepsilon}{2})_+$ in place of a, $g_{0,\frac{\varepsilon}{2}}(a)$ in place of e, and $\frac{\varepsilon}{2}$ in place of ε. Then, with the resulting c.p.c. order zero map ϕ, we can see that $b := \phi(e_{11})$ satisfies $k[b] \leqslant [a]$ and $[(a-\varepsilon)_+] \leqslant (k+1)(N+1)[b]$. □

Proposition 2.2. *The properties of M-comparison and N-almost divisibility pass to quotients and products of* C**-algebras (and in particular, they pass to ultraproducts). More specifically, given* C**-algebras A and $(A_\lambda)_{\lambda\in\Lambda}$, if they all have either one of these properties then so do $\prod_{\lambda\in\Lambda} A_\lambda$ and A/I for any closed two-sided ideal $I \subseteq A$.*

Proof. It is shown in [17, Lemma 2.3] that property of being unperforated passes to quotients and products. The same proof, with minor modifications, applies to M-comparison. A key fact is that $(\prod_\lambda A_\lambda) \otimes \mathcal{K}$ is a hereditary subalgebra of $\prod_\lambda (A_\lambda \otimes \mathcal{K})$; this is true because

$$(\prod_\lambda A_\lambda) \otimes M_n(\mathbb{C}) = \prod_\lambda (A_\lambda \otimes M_n(\mathbb{C}))$$

is a hereditary subalgebra of $\prod_\lambda (A_\lambda \otimes \mathcal{K})$ for each n (where we are viewing $M_n(\mathbb{C})$ as a corner of $\mathcal{K}$).

As for N-almost-divisibility, it is clear that the condition in Lemma 2.1 (ii) passes to products and quotients (cf. [18, Proposition 8.4], where divisibility of the unit is shown to pass to sequences). □

2.4. **Nuclear dimension.** Let A and B be C*-algebras. Let $\phi\colon A \to B$ be a completely positive contractive (c.p.c.) map. Let us say that ϕ has **order zero** if it preserves orthogonality, i.e., $ab = 0$ implies $\phi(a)\phi(b) = 0$ for all $a, b \in A$. By [38, Theorem 2.3], any such map has the form $\phi(a) = h\pi_\phi(a)$, where $\pi_\phi\colon A \to M(C^*(\phi(A))$ is a homomorphism, and $h \in M(C^*(\phi(A)))$ commutes with $\phi(A)$. We will make use of the functional calculus on order zero maps introduced by Winter and Zacharias: if the function $f \in C_0(0, \|\phi\|]$ is positive and of norm at most 1, then we set $f(\phi) := f(h)\pi_\phi$, which is also a c.p.c. map of order zero from A to B, and it satisfies $f(\phi)(p) = f(\phi(p))$ for every projection $p \in A$.

Following Winter and Zacharias [39], we say that a C*-algebra A has **nuclear dimension** at most m if for each finite set $F \subset A$ and $\varepsilon > 0$ there exist c.p.c. maps

$$A \xrightarrow{\psi_k} C_k \xrightarrow{\phi_k} A$$

with $k = 0, 1, \ldots, m$, such that ϕ_k is an order zero map for all k and

$$a \approx_\varepsilon \sum_{k=0}^{m} \phi_k\psi_k$$

for all $a \in F$.

Recall that a C*-algebra of finite nuclear dimension has the m-comparison property. We point out the following consequence of the m-comparison property (proven in [39, Theorem 5.4] by different means):

Proposition 2.3. *If a* C*-*algebra A is simple, of finite nuclear dimension, and traceless, then A is purely infinite.*

Proof. Let A be traceless and of finite nuclear dimension. By the m-comparison property we have that $m[a]$ is properly infinite for any non-zero $[a] \in \mathrm{Cu}(A)$. But since A is simple and non-type I, by Glimm's Halving Lemma [16, Lemma 6.7.1], we have that for any non-zero $[b]$ there exists a non-zero $[a]$ such that $m[a] \leqslant [b]$, whence $[b]$ is properly infinite. It follows that A is purely infinite. □

2.5. **The Jiang-Su algebra.** Let us denote by $\mathcal{Z}_{k-1,k}$, with $k \in \mathbb{N}$, the prime dimension drop C*-algebras and by $\mathcal{Z}$ the Jiang-Su algebra.

A C*-algebra A is called $\mathcal{Z}$-stable or tensorially $\mathcal{Z}$-absorbing if $A \cong A \otimes \mathcal{Z}$. If A is separable, this is equivalent to having a unital embedding of $\mathcal{Z}$ in $\mathrm{F}(A)$ (see [22, Theorem 7.2.2]). In fact, by [14, Proposition 5.1] (cf. [31, Proposition 2.2]), it suffices to find unital embeddings of the dimension drop C*-algebras $\mathcal{Z}_{k-1,k}$ into $\mathrm{F}(A)$ for all $k \in \mathbb{N}$. Furthermore, Rørdam and Winter showed in [23, Proposition 5.1], that in order to have one such embedding it suffices to find a c.p.c. order zero map from $M_{k-1}(\mathbb{C})$ into $\mathrm{F}(A)$ with "small defect". Thus, we arrive at the following $\mathcal{Z}$-stability criterion:

Proposition 2.4 (cf. [37, Proposition 1.14])**.** *Let A be a separable* C*-*algebra. Then A is $\mathcal{Z}$-stable if and only if for each $k \in \mathbb{N}$ there exists a c.p.c. map of order zero $\phi\colon M_k(\mathbb{C}) \to \mathrm{F}(A)$ such that $[1 - \phi(1)] \ll [\phi(e_{11})]$ in the Cuntz semigroup of* $\mathrm{F}(A)$.

3. Divisibility for C*-algebras of finite nuclear dimension

In this section, we prove the following:

Theorem 3.1. *Given $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that the following holds: If A is a* C*-*algebra of nuclear dimension m, with no elementary subquotients and no simple purely infinite subquotients, then A is (m, N)-pure.*

That A has m-comparison has already been shown, by the first-named author in [17], so what is really proven here is N-almost-divisibility. This will be deduced from a quantitative analysis of the relation between the size of the finite dimensional representations of A and divisibility properties (in the Cuntz semigroup) of a strictly positive element in A. It is likely that the same result holds after dropping the finiteness condition of no simple purely infinite subquotients, but our present methods – specifically, the construction of almost full orthogonal elements in Lemma 3.4 – require it, as demonstrated in Example 3.5. Notice that if A has finite decomposition rank then it satisfies this condition, since its simple subquotients also have finite decomposition rank and thus cannot be purely infinite.

Proposition 3.2. *Let $m, k \in \mathbb{N}$. Let A be a* C*-*algebra of nuclear dimension m and such that every representation of A has dimension at least k.*

(i) For each $\varepsilon > 0$ and strictly positive $c \in A_+$ there exist c.p.c. maps of order zero $\phi^j \colon M_k(\mathbb{C}) \to A$, with $j = 1, 2, \ldots, 2(m+1)$, such that

$$[(c-\varepsilon)_+] \leqslant \Big[\sum_{j=1}^{2(m+1)} \phi^j(1)\Big].$$

(ii) For each $\varepsilon > 0$ and strictly positive $c \in A_+$ there exists $a \in A_+$ such that

$$[(c-\varepsilon)_+] \leqslant k[a] \leqslant 2(m+1)[c].$$

Proof. (i): Let $A \xrightarrow{\psi^j} F_j \xrightarrow{\phi^j} A$, with $j = 0, 1, \ldots, m$, be an approximate factorization of id_A, where the maps ϕ^j are c.p.c. of order zero, the algebras F_j are finite dimensional, and

$$\sum_{j=0}^{m} \phi^j \psi^j(c) \approx_\varepsilon c.$$

Then $[(c-\varepsilon)_+] \leqslant \sum_{j=0}^{m} [\phi^j(1_{F_j})]$. If every representation of A has dimension at least k, we may assume that the matrix sizes of every matrix summand of each F_j are all at least k (by [39, Proposition 3.4]). This implies that for each j there exist c.p.c. maps of order zero $\phi^j_1, \phi^j_2 \colon M_k(\mathbb{C}) \to F_j$ such that $[1_{F_j}] \leqslant [\phi^j_1(1)] + [\phi^j_2(1)]$. The collection of maps ϕ^j_i, with $j = 0, 1, \ldots, m$ and $i = 0, 1$ has the desired properties.

(ii): Simply set $a := \sum_{j=1}^{2(m+1)} \phi^j(e_{11})$, with $\phi^j \colon M_k(\mathbb{C}) \to A$ as in part (i). The desired properties for a are readily verified. □

Lemma 3.3. *Let A be a* C*-*algebra with finite nuclear dimension and no simple purely infinite quotients. Then neither the Cuntz semigroup of A nor of its quotients can contain a full, compact, and properly infinite element.*

Proof. Let us argue by contradiction. Stabilizing and passing to a quotient of A if necessary, let us assume that there exists a full element $a \in A_+$ such that $[a] \ll [a]$ and $2[a] = [a]$. Let $\varepsilon > 0$ be such that $[(a-\varepsilon)_+] = [a]$. Since Cuntz equivalent elements generate the same ideal, $(a-\varepsilon)_+$ is also full. It follows that there exists at least one proper maximal ideal I of A. Then A/I is a simple C*-algebra of finite nuclear dimension containing a compact, stably properly infinite, positive element. By Proposition 2.3, this C*-algebra is purely infinite, which contradicts our hypotheses. □

The next lemma deals with the construction of full orthogonal elements. The construction is essentially the same one pioneered by Winter in [35, Proposition 3.6].

Lemma 3.4. *Given $m, l \in \mathbb{N}$ there exist $K, L > 0$ with the following property: If A is a* C**-algebra of nuclear dimension at most m, such that every representation has dimension at least K, and A has no simple purely infinite quotients, then for each $\varepsilon > 0$ and strictly positive element $c \in A_+$ there exist mutually orthogonal elements $d^0, d^1, d^2, \ldots, d^l \in A_+$ such that $[(c-\varepsilon)_+] \leqslant L[d^i]$ for $i = 0, 1, \ldots, l$.*

Proof. Let us first deal with the case $l = 1$. Let A be as in the statement. (The values of K and L will be specified in the argument that follows.) Let $\varepsilon > 0$ and let $c \in A_+$ be strictly positive. By the Proposition 3.2 (ii), if $K \geqslant 2m+3$ then there exists $a \in A_+$ such that

$$[\left(c - \frac{\varepsilon}{2}\right)_+] \leqslant (2m+3)[a] \leqslant 2(m+1)[c].$$

Let $\delta > 0$ be such that $[(c-\varepsilon)_+] \leqslant (2m+3)[(a-\delta)_+]$. Let us define

$$\begin{aligned} d^0 &= g_\delta(a), \\ d^1 &= (1 - g_{\frac{\delta}{2}}(a))^{\frac{1}{2}} c (1 - g_{\frac{\delta}{2}}(a))^{\frac{1}{2}}. \end{aligned}$$

It is clear that d^0 and d^1 are orthogonal and that $[(c-\varepsilon)_+] \leqslant (2m+3)[d^0]$. As for d^1, we have that

$$[c] \leqslant [g_{\frac{\delta}{2}}(a)] + [d^1]. \tag{3.1}$$

Let $\bar{\varepsilon} > 0$ be such that $(2m+3)[g_{\frac{\delta}{2}}(a)] \leqslant (2m+2)[(c-\bar{\varepsilon})_+]$. Multiplying by $2m+3$ in (3.1) we get

$$(2m+3)[c] \leqslant (2m+2)[(c-\bar{\varepsilon})_+] + (2m+3)[d^1]. \tag{3.2}$$

Let us show that d^1 is full in A. Let I be the closed two-sided ideal generated by d^1. Passing to the quotient by I in (3.2) we get $(2m+3)[c_I] \leqslant (2m+2)[(\pi_I(c)-\bar{\varepsilon})_+]$. Thus, $(2m+2)[\pi_I(c)]$ is properly infinite and compact. By the previous lemma, $\pi_I(c) = 0$; i.e., d^1 is full.

Since d^1 is full, a finite multiple of $[d^1]$ majorizes $[g_{\frac{\delta}{2}}(a)]$. Thus, by (3.1), a finite multiple of $[d^1]$ majorizes $[c]$. Now from (3.2) we deduce that $d_\tau(c) \leqslant (2m+3)d_\tau(d^1)$ for all $\tau \in \mathrm{T}(A)$. By the m-comparison property this implies that $[c] \leqslant 2(m+1)(2m+3)[d^1]$. This completes the proof for $l = 1$.

For the general case we proceed by induction. From the relation $[c] \leqslant 2(m+1)(2m+3)[d^1]$ we deduce that if all the representations of A have large enough dimension, then so do the representations of $\mathrm{her}(d_1)$ (in a way that depends only on m). Thus, we can apply the induction hypothesis to the hereditary subalgebra generated by d^1. □

Example 3.5. The construction of full orthogonal elements in Lemma 3.4 uses the fact that c from Proposition 3.2 has small trace, so that under the right finiteness conditions, $1 - g_\varepsilon(c)$ is full. However, if A is simple, unital, and purely infinite, then (for any k) there are c.p.c. order zero maps ϕ^j, for $j = 1, 2$ satisfying (i) of Proposition 3.2, with

$$\phi^1(e_{11}) + \phi^2(e_{11}) = 1$$

(it is enough to get these maps into $\mathcal{O}_2$, which is easy.) Using such maps, the construction of c in the proof of Proposition 3.2 then yields $c = 1$, so that there is no way to use functional calculus on c to produce full orthogonal elements.

This demonstrates that an entirely different approach to constructing full orthogonal elements is needed to go beyond situations where finiteness conditions are assumed. This problem is also present in the argument in [37].

Lemma 3.6. *Given $m \in \mathbb{N}$ there exist $M, N > 0$ with the following property: If $k \in \mathbb{N}$ and A is a* C*-*algebra of nuclear dimension at most m, such that every representation has dimension at least $k \cdot M$, and A has no simple purely infinite quotients, then for each $\varepsilon > 0$ and strictly positive $c \in A_+$ there exists $b \in A_+$ such that $k[b] \leqslant [c]$ and $[(c-\varepsilon)_+] \leqslant kN[b]$.*

Proof. Let K and L be constants as in the previous proposition corresponding to $l := m+1$. By Proposition 3.2 (ii), if every representation of A has dimension at least $2k(m+1)L$, then there exists $a \in A_+$ such that

$$[(c - \frac{\varepsilon}{2})_+] \leqslant 2kL(m+1)[a] \leqslant 2(m+1)[c].$$

Let us choose $\delta_1 > 0$ first, and then $\delta_2 > 0$, such that

$$[(c-\varepsilon)_+] \leqslant 2kL(m+1)[(a-\delta_1)_+] \leqslant 2(m+1)[(c-\delta_2)_+].$$

If every representation of A has dimension at least K, then there exist mutually orthogonal elements $d^0, d^1, \dots, d^m \in A_+$ such that $[(c-\delta_2)_+] \leqslant L[d^i]$ for all i. It follows that

$$2kL(m+1)[(a-\delta_1)_+] \leqslant 2L(m+1)[d^i]$$

for all i. Thus, by the m-comparison property

$$k[(a-\delta_1)_+] \leqslant \sum_{i=0}^{m} [d^i] \leqslant [c].$$

Therefore, setting $b := (a-\delta_1)_+$, $M := \max(K, 2L(m+1))$, and $N := 2L(m+1)$ (both of which only depend on m), we get the desired result. □

Proof of Theorem 3.1. By [17], A has m-comparison. Let $N > 0$ be as in the previous lemma. Since no subquotient of A is elementary, for each $a \in (A \otimes \mathcal{K})_+$ the C*-algebra her(a) has no finite dimensional representations. So the previous lemma is applicable to her(a) and any $k \in \mathbb{N}$, whence showing that A is N-almost divisible. □

Let us say that the C*-algebra A has strong tracial M-comparison if for all $[a], [b] \in \mathrm{Cu}(A)$, we have that $[a] <_s \frac{1}{M}[b]$ implies that $[a] \leqslant [b]$.

Theorem 3.7. *Let $m \in \mathbb{N}$. There exists $M > 0$ such that if A is a* C*-*algebra of nuclear dimension at most m with no simple purely infinite subquotients then A has strong tracial M-comparison.*

Proof. This argument is akin to an argument in [37, Section 3], that (M,N)-pureness implies strong tracial $\overline{M}$-comparison, for some $\overline{M}$. However, extra steps are taken here, to avoid assuming that A has no elementary subquotients.

Say $Md_\tau(a) \leqslant d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$ (how large M should be will be specified later). Letting τ be a trace that is 0 on a closed two-sided ideal and ∞ outside, we conclude that the ideal generated by a is contained in the ideal generated by b. We may reduce to the case that b generates the same ideal as a. To see this, let $e_0 \in (A \otimes \mathcal{K})_+$ be a strictly positive element of the ideal generated by a. Let $\bar{b} = e_0 b e_0$. Then $[\bar{b}] \leqslant [b]$ and $Md_\tau(a) \leqslant d_\tau(e_0 b e_0)$ for all τ.

So let us assume that a and b generate the same ideal. We claim that each representation of her(b) has dimension at least M. Indeed, no such representation, after being extended to the ideal generated by b, can vanish on a (since a is a full element of this ideal). Then $Md_\tau(a) \leqslant d_\tau(b)$ implies the claim.

Let $\varepsilon > 0$ and choose $\delta > 0$ such that $Md_\tau((a-\varepsilon)_+) \leqslant d_\tau((b-\delta)_+)$ for all $\tau \in \mathrm{T}(A)$. If M is large enough (depending only on m), there exist – by Lemma 3.4 – mutually orthogonal positive elements $d^0, d^1, \dots, d^m \in$ her(b) such that $[(b-\delta)_+] \leqslant L[d^i]$ for all i and some $L > 0$.

Thus, $Md_\tau((a-\varepsilon)_+) \leqslant L[d^i]$ for all i and $\tau \in \mathrm{T}(A)$. Again, if M is large enough (relative to L, which again depends only on m), then by m-comparison we conclude that

$$[(a-\varepsilon)_+] \leqslant \sum_{i=0}^{m} [d_i] \leqslant [b].$$

Since $\varepsilon > 0$ can be arbitrarily small, we get $[a] \leqslant [b]$, as desired. □

3.1. **$\mathcal{Z}$-stability of infinite tensor products.** In [4], Dadarlat and Toms showed that if a unital C*-algebra A admits a unital embedding of an approximately subhomogeneous C*-algebra without 1-dimensional representations, then $\bigotimes_{n=1}^{\infty} A$ is $\mathcal{Z}$-stable. As shown in [4, 6.3], this question quickly reduces to the case that A is an RSH algebra with finite topological dimension and without 1-dimensional representations. The proof in [4] then relies on sophisticated tools from homotopy theory. We give here a more abstract proof of Dadarlat and Toms's result using the results on divisibility previously obtained in this section. We prove the following:

Theorem 3.8. *Let A be a separable unital* C*-*algebra such that*

(i) A has no 1-dimensional representations,

(ii) A satisfies that

$$\frac{\dim_{\mathrm{nuc}}(A^{\otimes n})}{\alpha^n} \to 0, \tag{3.3}$$

for any $\alpha > 1$,

(iii) for all n, no simple quotient of $A^{\otimes n}$ is purely infinite (e.g., if A has finite decomposition rank).

Then $A^{\otimes\infty}$ is $\mathcal{Z}$-stable. More generally, the same conclusion holds if $A^{\otimes\infty}$ admits a unital embedding of a C*-*algebra with these properties.*

Conditions (ii) and (iii) above are satisfied if A is an RSH algebra of finite topological dimension. Indeed, by [34, Theorem 1.6], in this case A has finite decompoition rank and $\dim_{\mathrm{nuc}}(A^{\otimes n})$ has linear growth. In this way we recover Dadarlat and Toms's result.

Although we will not use any of the results in this section in the sequel, many of the ideas encountered here will reappear. A simplification here is that it is easy to arrange commutativity in $A^{\otimes\infty}$.

For the remainder of this section, we let A denote a separable unital C*-algebra that satisfies (i)-(iii) of the above theorem.

Lemma 3.9. *There exists k such that $A^{\otimes k}$ has two full orthogonal elements.*

Proof. By the proof of Lemma 3.4, if a unital C*-algebra B has no simple purely infinite quotients and all its representations have dimension at least $2\dim_{\mathrm{nuc}}(B)+3$ then B contains two full orthogonal elements. But all the representations of $A^{\otimes k}$ have dimension at least 2^k, which, by (3.3), majorizes $2\dim_{\mathrm{nuc}}(A^{\otimes k})+3$ for k large enough. Thus, the result follows. □

Lemma 3.10. *For all $q \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that there exists an order zero map $\phi\colon M_q(\mathbb{C}) \to A^{\otimes k}$ whose image is full.*

Proof. We may assume without loss of generality that $q = 2^n$ for some $n \in \mathbb{N}$. Replacing A by $A^{\otimes k}$, with k as in the previous lemma, we may also assume that A contains two full orthogonal elements.

Let $\gamma_n = \dim_{\mathrm{nuc}}(A^{\otimes n})$. Since every representation of $A^{\otimes n}$ has dimension at least 2^n, there exist order zero maps $\psi_i\colon M_{2^n}(\mathbb{C}) \to A^{\otimes n}$, with $i = 1, 2, \ldots, 2(\gamma_n+1)$ such that $[1] \leqslant$

$\sum_{i=1}^{2(\gamma_n+1)}[\psi_i(1)]$ (by Proposition 3.2). On the other hand, since A contains two full positive orthogonal elements, $A^{\otimes m}$ contains 2^m full and pairwise orthogonal positive elements for all $m \in \mathbb{N}$. Let us choose m large enough such that $2^m \geqslant 2(\gamma_n+1)$ and let us denote these orthogonal elements by $d_0, d_1, \ldots, d_{2^m} \in A^{\otimes m}$. Let us define $\phi\colon M_{2^n}(\mathbb{C}) \to A^{\otimes n} \otimes A^{\otimes m}$ by

$$\phi = \sum_{i=1}^{2(\gamma_n+1)} \phi_i \otimes d_i.$$

It can be readily verified that ϕ has the desired properties. □

Proof of Theorem 3.8. By Proposition 2.4, we must construct for each $q \in \mathbb{N}$ a c.p.c. map of order zero $\phi\colon M_q(\mathbb{C}) \to \mathrm{F}(A^{\otimes\infty})$ such that $[1-\phi(1)] \ll [\phi(e_{11})]$. In fact, it suffices to construct one such map ϕ from $M_q(\mathbb{C})$ into $A^{\otimes\infty}$ (by then considering the central sequence of maps $\phi\otimes 1\otimes\cdots$, $1\otimes\phi\otimes 1\otimes\cdots$, etc, from $M_q(\mathbb{C})$ to $A^{\otimes\infty}\otimes A^{\otimes\infty}\otimes\cdots\cong A^{\otimes\infty}$). Let us do this.

Let A be a C*-algebra that satisfies conditions (1)-(3) of the theorem. By the previous lemma, we may assume that there exists $\psi\colon M_q(\mathbb{C}) \to A$ such that $\psi(1)$ is full, i.e., $[1] \leqslant Q[\psi(1)]$, with $Q > 0$. Using functional calculus on the order zero map ψ, we may also assume that $2Q[1-\psi(1)] \leqslant (2Q-1)[1]$ (see the proof of Lemma 6.10 below). Let $\varepsilon > 0$ be such that $[1] \leqslant Q[(\psi(1)-\varepsilon)_+]$.

Let $n \in \mathbb{N}$. Let $\psi_i\colon M_q(\mathbb{C}) \to A^{\otimes n}$, with $i = 1, 2, \ldots, n$ be given by $\psi_i = 1\otimes\cdots\otimes\psi\otimes\cdots\otimes 1$. By Lemma 6.9 (ii) (essentially, Winter's [36, Lemma 2.3]), there exists a c.p.c. map of order zero $\phi\colon M_q(\mathbb{C}) \to A^{\otimes n}$ such that $\psi_1 \leqslant \phi$ and

$$1-\phi(1) = \prod_{i=1}^{n}(1-\psi_i(1)) = \bigotimes_{i=1}^{n}(1-\psi(1)).$$

Thus, we find that

$$(2Q)^n[1-\phi(1)] \leqslant (2Q-1)^n[1].$$

Let $\gamma_n = \dim_{\mathrm{nuc}}(A^{\otimes n})$. Let $d^0, d^1 \in A_+$ be orthogonal and such that $[1] \leqslant L[d^i]$ for some $L > 0$ and $i = 0, 1$. Set $m = \lceil \log_2(\gamma_n+1)\rceil$. In $A^{\otimes m}$ we can find 2^m (approximately γ_n+1) positive orthogonal elements $d_1, d_2, \ldots, d_{2^m}$ such that $[1] \leqslant L^m[d_i]$ for all i. Let us choose $m = \lceil \log_2(\gamma_n+1)\rceil$ (so that there are approximately γ_n+1 orthogonal elements). Notice that $m < n$ for n lare enough by (3.3). Let us regard $A^{\otimes m}$ as a subalgebra of $1\otimes A^{\otimes n-1}$. Then,

$$\begin{aligned}(2Q)^n[1-\phi(1)] &\leqslant (2Q-1)^n[1]\\ &\leqslant (2Q-1)^n Qq[\psi_1((e_{11}-\varepsilon)_+]\\ &\leqslant (2Q-1)^n QqL^m[\psi((e_{11}-\varepsilon))_+\otimes d_i]\end{aligned}$$

for all $i = 1, \ldots, 2^m$. We claim that

$$\frac{(2Q-1)^n QqL^m}{(2Q)^n} \to 0. \tag{3.4}$$

Indeed, notice that $L^m = L^{\lceil \log_2(\gamma_n+1)\rceil} = O(\gamma_n^{\log_2 L})$, so (3.4) follows from (3.3). Thus, choosing n large enough, we get that

$$[1-\phi(1)] <_s [\psi((e_{11}-\varepsilon))_+\otimes d_i]$$

for all $i = 1, 2, \dots, 2^m$. By the γ_n-comparison property in $A^{\otimes n}$ [17], we conclude that

$$[1-\phi(1)] \leqslant \sum_{i=1}^{\gamma_n+1} [\psi((e_{11}-\varepsilon))_+ \otimes d_i] \leqslant [\psi((e_{11}-\varepsilon))_+ \otimes 1] \ll [\phi(e_{11})].$$

This completes the proof. □

4. Central factorization

A powerful way to use finite nuclear dimension (in the separable case) is via an exact factorization of the canonical embedding $A \hookrightarrow A_\omega$ using order zero maps into ultraproducts of finite dimensional C*-algebras (as proven in [17, Proposition 2.2], using [39, Proposition 3.2]). Here, we show that a similar factorization for $\mathrm{F}(B, A)$ may be made when $B \subset A_\omega$ is a separable C*-subalgebra of finite nuclear dimension. The finite dimensional C*-algebras in the ultraproducts, however, become replaced by direct sums of hereditary subalgebras of A. This factorization result can (and will) be applied to push certain regularity properties of A to $\mathrm{F}(B, A)$ (just as 0-comparison for finite dimensional C*-algebras gets pushed to m-comparison for a C*-algebra, by the first-named author in [17]). Before stating the factorization result, we introduce notation.

Let A be a C*-algebra. Let $c \in M(A)_+$ be a contraction. Define the c.p.c. map $q_c\colon A \to \overline{cAc}$ given by $q_c(x) = c^{\frac{1}{2}}xc^{\frac{1}{2}}$. If $\Sigma \subset M(A)_+$ is a finite set of positive contractions then we define $\mathbf{C}_\Sigma := \bigoplus_{c\in\Sigma} \overline{cAc}$ and $\mathbf{Q}_\Sigma\colon A \to \mathbf{C}_\Sigma$ by

$$\mathbf{Q}_\Sigma = \bigoplus_{c\in\Sigma} q_c. \tag{4.1}$$

We may write $\mathbf{C}^A_\Sigma$ and $\mathbf{Q}^A_\Sigma$ if there is ambiguity in the choice of the ambient C*-algebra.

For a sequence of finite sets $\Sigma_n \subset M(A)_+$ of positive contractions, define

$$\mathbf{C}^A_{(\Sigma_n)_n} := \prod_\omega \mathbf{C}^A_{\Sigma_n}$$

and set

$$\mathbf{Q}^A_{(\Sigma_n)_n} := \pi_\omega \circ (\mathbf{Q}^A_{\Sigma_1}, \mathbf{Q}^A_{\Sigma_2}, \cdots)\colon A \to \mathbf{C}^A_{(\Sigma_n)_n}$$

Now, suppose that $B \subseteq A$ and we have a sequence of finite sets $\Sigma_n \subset B$. Then the restriction of $\mathbf{Q}^A_{(\Sigma_n)_n}$ to $A \cap B'$ is of order zero, and factors through $(A \cap B')/B^\perp$. Let us denote by $\widetilde{\mathbf{Q}}_{(\Sigma_n)_n}\colon (A\cap B')/B^\perp \to \mathbf{C}_{(\Sigma_n)_n}$ the factor map.

Here is the main result to be proven in this section.

Theorem 4.1. *Let A be a C*-algebra and let $B \subset A$ be a separable C*-subalgebra of nuclear dimension m. For each $k = 0, 1, \dots, 2m+1$ there exist maps*

$$(A\cap B')/B^\perp \xrightarrow{Q_k} C_k \xrightarrow{R_k} (A_\omega \cap B')/B^\perp,$$

such that

(i) *For each k, there exists a sequence $(\Sigma^k_n)_{n=1}^\infty$, where each $\Sigma^k_n \subset B$ is a finite set of positive contractions, such that $C_k = \mathbf{C}_{(\Sigma^k_n)_n}$ and $Q_k = \widetilde{\mathbf{Q}}_{(\Sigma^k_n)_n}$. In particular, Q_k is a c.p.c. map of order zero.*

(ii) *For each k, R_k is a c.p.c. map of order zero.*

(iii) For all $a \in (A \cap B')/B^\perp$ we have

$$a = \sum_{k=0}^{2m+1} R_k Q_k(a).$$

Remark 4.2. Suppose that A is an ultraproduct algebra. Then, given C_k, Q_k, R_k as in Theorem 4.1, we may improve our lot somewhat, using the diagonal sequence argument (cf. [27, Section 4.1]) as follows. Given a separable subset D of $(A\cap B')/B^\perp$ and for each k, a separable subset C_k' of C_k containing $Q_k(D)$, there exist *-linear maps

$$\hat{R}_k \colon C_k \to (A \cap B')/B^\perp$$

such that:

(i) $\hat{R}_k|_{C_k'}$ is c.p.c. order zero, and
(ii) $a = \sum_{k=0}^{2m+1} \hat{R}_k Q_k(a)$ for all $a \in D$.

The maps R_k in the above theorem come chiefly from maps χ_ϕ that we define presently. Let $\phi\colon M_p(\mathbb{C}) \to A$ be a c.p.c. map of order zero map and set $c = \phi(e_{11})$. Let us define a homomorphism $\chi_\phi\colon \mathrm{her}(c) \to A$ by

$$\chi_\phi(x) = \sum_{i=1}^{p} \pi_\phi(e_{i1}) x \pi_\phi(e_{1i}).$$

Lemma 4.3. *Let $\phi\colon M_p(\mathbb{C}) \to A$ be a c.p.c. map of order zero and let c, χ_ϕ be as defined above. For each contraction $a \in A$ we have*

$$\|[a, \phi^{1/2}]\| < \varepsilon \Rightarrow \chi_\phi q_c(a) \approx_{3\varepsilon} \phi(1)a. \tag{4.2}$$

Proof. We have

$$\begin{aligned} \phi(1)a &= \int_{u \in U(M_p(\mathbb{C}))} \phi^{1/2}(u)^* \phi^{1/2}(u) a \, du \\ &\approx_\varepsilon \int_{u \in U(M_p(\mathbb{C}))} \phi^{1/2}(u)^* a \phi^{1/2}(u) \, du \\ &= \frac{1}{p} \sum_{i,j=1}^{p} \phi^{1/2}(e_{ij}) a \phi^{1/2}(e_{ji}). \qquad (4.3) \end{aligned}$$

Now, note that, for $\eta > 0$,

$$\begin{aligned} \phi^{1/2}(e_{ij})a\phi^{1/2}(e_{ji}) &\approx_{\eta^{1/2}} g_{0,\eta}(\phi)(e_{i1})\phi^{1/2}(e_{1j})a\phi^{1/2}(e_{ji}) \\ &\approx_\varepsilon g_{0,\eta}(\phi)(e_{i1})a\phi^{1/2}(e_{1j})\phi^{1/2}(e_{ji}) \\ &= g_{0,\eta}(\phi)(e_{i1})a\phi^{1/2}(e_{11})\phi^{1/2}(e_{1i}) \\ &\approx_\varepsilon g_{0,\eta}(\phi)(e_{i1})\phi^{1/2}(e_{11})a\phi^{1/2}(e_{1j}) \\ &\approx_{\eta^{1/2}} \phi^{1/2}(e_{i1})a\phi^{1/2}(e_{1i}), \end{aligned}$$

and since η is arbitrary,

$$\phi^{1/2}(e_{ij})a\phi^{1/2}(e_{ji}) \approx_{2\varepsilon} \phi^{1/2}(e_{i1})a\phi^{1/2}(e_{1i}).$$

It follows that, for each i,

$$\frac{1}{p}\sum_j \phi^{1/2}(e_{ij})a\phi^{1/2}(e_{ji}) \approx_{2\varepsilon} \phi^{1/2}(e_{i1})a\phi^{1/2}(e_{1i}).$$

Finally, by orthogonality of the errors, it follows that

$$\frac{1}{p}\sum_{i,j} \phi^{1/2}(e_{ij})a\phi^{1/2}(e_{ji}) \approx_{2\varepsilon} \sum_i \phi^{1/2}(e_{i1})a\phi^{1/2}(e_{1i}). \tag{4.4}$$

Combining (4.3) and (4.4) produces (4.2). □

Remark. It is simpler to show that

$$\|[a,\phi]\| < \varepsilon \Rightarrow \chi_\phi q_c(a) \approx_{p\varepsilon} \phi(1)a,$$

and this estimate (differing in that the approximation on the left depends on the matrix size p) ultimately suffices for our application. Nevertheless, the stronger estimate seems independently interesting.

Lemma 4.4. *Given $f \in C_0((0,1])_+$ and $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds: If $\beta\colon D \to A$ is a c.p.c. order zero map between* C*-*algebras D and A, where D is unital, and $a \in A$ is a contraction which satisfies*

$$\|[a,\beta]\| < \delta$$

then

$$\|[a,f(\beta)]\| < \varepsilon.$$

Proof. Let $g \in C_0((0,1])_+$ be such that $f \approx_{\varepsilon/4} g \cdot \mathrm{id}_{[0,1]}$. Then (by approximating g by polynomials), we may find $0 < \delta < \frac{\varepsilon}{4\|g\|}$ such that, if a,b are elements of a C*-algebra such that b is a positive contraction and $\|[a,b]\| < \delta$ then $\|[a,g(b)]\| < \varepsilon/4$.

Now, suppose that we have β and a as in the statement of the lemma. We compute, for a contraction $x \in D$,

$$\begin{aligned} af(\beta)(x) &\approx_{\frac{\varepsilon}{4}} ag(\beta(1))\beta(x) \\ &\approx_{\frac{\varepsilon}{4}} g(\beta(1))a\beta(x) \\ &\approx_{\|g\|\frac{\varepsilon}{4\|g\|}} g(\beta(1))\beta(x)a \\ &\approx_{\frac{\varepsilon}{4}} f(\beta)(x)a. \end{aligned}$$

□

The proof of the following lemma contains the basic construction upon which the other results will be built.

Lemma 4.5. *Let D be a finite dimensional* C*-*algebra. For each $\varepsilon > 0$ there exists $\delta > 0$ with the following property:*

If A is a C*-*algebra and $\beta\colon D \to A$ is a c.p.c. map of order zero, then there exist maps*

$$A \xrightarrow{Q_k} C_k \xhookrightarrow{R_k} A$$

with $k = 0,1$, with the following properties:

- *(i) For each $k = 0,1$, there exists a finite set of positive contractions $\Sigma_k \subset C^*(\mathrm{im}(\beta))^\sim$ such that $C_k = \mathbf{C}^A_{\Sigma_k}$ and $Q_k = \mathbf{Q}^A_{\Sigma_k}$ (defined as in (4.1)).*
- *(ii) For each $k = 0,1$, R_k is an injective *-homomorphism. Furthermore, there exists $h_k \in C_0((0,1])_+$ $\|h_k - \mathrm{id}_{[0,1]}\| < \varepsilon$ and such that $[R_k, h_k(\beta)] = 0$.*

(iii) If $a \in A$ is a contraction such that $\|[a,\beta]\| < \delta$ then

$$R_0Q_0(a) + R_1Q_1(a) \approx_\varepsilon a.$$

(iv) $\|[R_k,\beta]\| < \varepsilon$.

Proof. Let us take a partition of unity $F = F_0 \amalg F_1$ for $C([0,1])$, consisting of positive elements whose supports each have diameter at most ε, and such that for each $k = 0,1$, the elements of F_k have pairwise disjoint (closed) supports. It follows that there exists $h_k \in C_0((0,1])^+$ which is constant on the support of each element of F_k, and such that

$$\|h_k - \mathrm{id}_{[0,1]}\| \leqslant \varepsilon.$$

Observe that, for $k = 0,1$ and $f \in F_k$, since h_k is constant on the support of f, for any positive contraction y in a C*-algebra (which below we will take to be $\beta(1)$), we have

$$[h_k(y), \mathrm{her}(f(y))] = 0. \tag{4.5}$$

We may assume, without loss of generality, that there exists $f_0 \in F_0$ such that $f_0(1) = 1$; therefore, $F \setminus \{f_0\} \subset C_0((0,1])$. Let $D = \bigoplus_{i=1}^q M_{n_i}(\mathbb{C})$. By Lemma 4.4, let $\delta > 0$ be such that, if β is a c.p.c. order zero map from a unital C*-algebra to a C*-algebra containing a contraction a, and $\|[a,\beta]\| < \delta$ then $\|[a, f(\beta)^{1/2}]\| < \frac{\varepsilon}{6q|F|}$ for all $f \in F\setminus\{f_0\}$, and additionally, $\|[a, f_0(\beta(1))^{1/2}]\| < \frac{\varepsilon}{2}$.

For each $i = 1,2,\ldots,q$ and $f \in F \setminus \{f_0\}$, define

$$\begin{aligned} \beta_{f,i} &:= f(\beta)|_{M_{n_i}(\mathbb{C})} \colon M_{n_i}(\mathbb{C}) \to A, \text{ and} \\ c_{f,i} &:= \beta_{f,i}(e_{11}) \in A^+. \end{aligned}$$

Set $c_0 := f_0(\beta(1)) \in A^\sim$. Define

$$\begin{aligned} \Sigma_0 &:= \{c_{f,i} \mid f \in F_0 \setminus \{f_0\}, i = 1,\ldots,q\} \cup \{c_0\}, \text{ and} \\ \Sigma_1 &:= \{c_{f,i} \mid f \in F_1, i = 1,\ldots,q\}. \end{aligned}$$

Let us define Q_0, Q_1, C_0, C_1 accordingly as in the statement of the lemma. Let us define $R_k \colon Q_k \to A$ by

$$\begin{aligned} R_0((b_c)_{c\in\Sigma_0}) &= \sum_{\substack{f\in F_0 \\ f\neq f_0}} \sum_{i=1}^q \chi_{\beta_{f,i}}(b_{c_{f,i}}) + b_{c_0}, \\ R_1((b_c)_{c\in\Sigma_1}) &= \sum_{f\in F_1} \sum_{i=1}^q \chi_{\beta_{f,i}}(b_{c_{f,i}}). \end{aligned}$$

Notice that each R_k is a homomorphism since it is a sum of homomorphisms with orthogonal ranges.

(i) clearly holds by construction. (ii) holds by (4.5).

To see (iii), let $a \in A$ be a contraction for which $\|[a,\beta]\| < \delta$. By our choice of δ using Lemma 4.4, it follows that, for $f \in F \setminus \{f_0\}$,

$$\|[a, f(\beta)^{1/2}]\| < \frac{\varepsilon}{6q|F|},$$

whence by Lemma 4.3,

$$\beta_{f,i}(1)a \approx_{\frac{\varepsilon}{2q|F|}} \chi_{\beta_{f,i}} q_{c_{f,i}}(a). \tag{4.6}$$

Also,

$$\|[a, f_0(\beta(1))^{1/2}]\| < \frac{\varepsilon}{2}. \tag{4.7}$$

We then compute

$$\begin{aligned} R_0Q_0(a) + R_1Q_1(a) &= \sum_{f\in F\setminus\{f_0\}} \sum_{i=1}^{q} \chi_{\beta_{f,i}} q_{c_{f,i}}(a) + c_0^{1/2} a c_0^{1/2} \\ &\approx_{\varepsilon}^{(4.6),(4.7)} \sum_{f\in F\setminus\{f_0\}} \sum_{i=1}^{q} \beta_{f_i}(1)a + f_0(\beta(1))(a) \\ &= \sum_{f\in F} f(\beta(1))a \\ &= a. \end{aligned}$$

(iv) follows from (ii), except with 2ε in place of ε. Therefore, using $\varepsilon/2$ instead of ε from the get-go will make (iv) hold as stated. ☐

Proposition 4.6. *Let A be a C^*-algebra and let $B \subseteq A$ be a C^*-subalgebra of nuclear dimension at most m. Then for each finite set $F \subset B$ and $\varepsilon > 0$ there exist a finite set $G \subset B$, $\delta > 0$, and maps*

$$A \xrightarrow{Q_k} C_k \xrightarrow{R_k} A,$$

with $k = 0, 1, \ldots, 2m+1$ such that

(i) For each k, there exists a finite set of positive contractions $\Sigma_k \subset (B^\sim)_+ \subseteq (A^\sim)_+$ such that $C_k = \mathbf{C}^A_{\Sigma_k}$ and $Q_k = \mathbf{Q}^A_{\Sigma_k}$ (as defined in (4.1)).

(ii) For each k, the map R_k is an order zero map and for every $f \in F$, we have $\|[f, R_k]\| < \varepsilon$.

(iii) If $a \in A$ is a contraction such that $\|[a, G]\| < \delta$ then

$$\sum_{k=0}^{2m+1} R_kQ_k(a) \approx_\varepsilon a.$$

Proof. Set $\eta := \varepsilon/(6m+5)$.

Let us find an approximation of the identity map on B within (F, η) by c.p.c. maps

$$B \xrightarrow{\alpha_k} D_k \xrightarrow{\beta_k} B,$$

with β_k of order zero and $k = 0, 1, \ldots, m$. Let e be a positive contraction which approximately acts as an identity on F. Set $e_k := \beta_k\alpha_k(e) \in B$ for $k = 0, 1 \ldots, m$ (the "partition of unity" of this decomposition). By [27, Lemma 3.4] (cf. [37, Proposition 4.2]), with an appropriate choice of e and of the decomposition, we have

$$\beta_k\alpha_k(a) \approx_\eta e_k a \ \forall k = 0, \ldots, m \quad \text{and} \quad a \approx_\eta \sum_{k=0}^{m} e_k a \tag{4.8}$$

for all $a \in F$.

Let us apply Lemma 4.5 to each order zero map β_k and with η in place of ε. We obtain maps

$$A \xrightarrow{Q_{k,j}} C_{k,j} \xrightarrow{R_{k,j}} A,$$

elements $h_{k,j} \in C_0((0,1])_+$ for $j = 0, 1$, and a number $\delta_k > 0$ satisfying (i)-(iv) of Lemma 4.5 for β_k and η. Let us define

$$\widetilde{R}_{k,j} := h_{k,j}(e_k) R_{k,j}. \tag{4.9}$$

Notice that, by Lemma 4.5 (ii), $h_{k,j}(e_k)$ commutes with $R_{k,j}$, and therefore $\widetilde{R}_{k,j}$ is an order zero map. Let us show that the data $Q_{k,j}$, $C_{k,j}$, and $\widetilde{R}_{k,j}$, with $k = 0, 1, \ldots, m$ and $j = 0, 1$, have the properties (i)-(iii) postulated by the proposition, for a suitable finite set $G \subset A$ and number $0 < \delta < \min_k \delta_k$ to be determined soon. (That is, the proposition as stated will follow by relabelling $(\widetilde{R}_{k,j})_{k=0,\ldots,m,j=0,1}$ to $(R_k)_{k=0,\ldots,2m+1}$.)

By Lemma 4.5 (i), (for $Q_{k,j}$, $C_{k,j}$, and $R_{k,j}$) property (i) is easily verified.

Let us show (iii). Let $a \in A$. Since the image of each β_k is finite-dimensional, we may find a finite subset G of A and a tolerance $\delta > 0$ such that $\|[a, G]\| < \delta$ implies that $\|[a, \beta_k]\|$ is sufficiently small, so that in turn by Lemma 4.5 (iii),

$$a \approx_\eta R_{k,0} Q_{k,0}(a) + R_{k,1} Q_{k,1}(a).$$

for all k. Thus, multiplying by $\tilde{e}_{k,j}$ on both sides and summing over k and j we get

$$\begin{aligned} a &\approx_\eta^{(4.8)} \sum_{k=0}^{m} e_k(a) \\ &\approx_{2(m+1)\eta} \sum_{k=0}^{m} \sum_{j=0,1} e_k R_{k,j} Q_{k,j}(a) \\ &\approx_{2(m+1)\eta}^{\text{Lemma 4.5 (ii)}} \sum_{k=0}^{m} \sum_{j=0,1} \tilde{R}_{k,j} Q_{k,j}(a). \end{aligned}$$

as desired (since $(4m + 5)\eta \leqslant \varepsilon$).

Finally, let us prove (ii). Let $f \in F$ and $b \in C_{k,j}$ be contractions. Then

$$\begin{aligned} f \cdot \widetilde{R}_{k,j}(b) &\approx_\eta^{(4.9)} f \cdot e_k \cdot R_{k,j}(b) \\ &\approx_\eta^{(4.8)} \beta_k \alpha_k(f) \cdot R_{k,j}(b) \\ &\approx_\eta^{\text{Lemma 4.5 (iv)}} R_{k,j}(b) \cdot \beta_k \alpha_k(f) \\ &\approx_{3\eta} \widetilde{R}_{k,j}(b) \cdot f, \end{aligned}$$

as desired (since $6\eta \leqslant \varepsilon$) □

The main theorem of this section will now be proven, essentially by turning approximate relations in the previous proposition, holding at the level of the algebra, into exact relations in the ultrapower algebra.

Proof of Theorem 4.1. Let $(F_n)_{n=1}^\infty$ be an increasing sequence of finite sets with dense union in B. For each F_n and with $\varepsilon_n = 1/n$, let us apply Proposition 4.6 to find $\delta_n > 0$, a finite $G_n \subset B$, finite sets $\Sigma_n^k \subset B^\sim$ with $k = 0, 1, \ldots, 2m + 1$, and c.p.c. maps of order zero $R_n^k \colon C_k \to A$ with $k = 0, 1, \ldots, 2m + 1$ that have the properties stated in the proposition. In particular, we have that

$$a \approx_{\frac{1}{n}} \sum_{k=0}^{2m+1} R_n^k Q_n^k(a)$$

for all $a \in A$ such that $\|[a, G_n]\| < \delta_n$. Drawing from an approximate identity, let $e_n \in B_+$ be such that $\|[e_n a e_n, G_n]\| < \delta_n$ for all contractions $a \in A \cap B'$ and $c^{\frac{1}{2}} e_n \approx_{\varepsilon_n} (c^{\frac{1}{2}} e_n^2 c^{\frac{1}{2}})^{\frac{1}{2}}$ for all $c \in \Sigma_n^k$ and for all k. Let $\widetilde{\Sigma}_n^k$ be the subset of B given by $\widetilde{\Sigma}_n^k := \{c^{\frac{1}{2}} e_n^2 c^{\frac{1}{2}} \mid c \in \Sigma_n^k\}$. Set

$$\begin{aligned} \widetilde{C}_n^k &:= \mathbf{C}_{\tilde{\Sigma}_n^k} \subseteq C_n^k \text{ and} \\ Q_n^k &:= \widetilde{\mathbf{Q}}_{(\Sigma_n^k)}. \end{aligned}$$

Then, for all contractions $a \in A \cap B'$,

$$e_n a e_n \approx_{\frac{1}{n}} \sum_{k=0}^{2m+1} R_n^k Q_n^k(e_n a e_n) \approx_{4(m+1)\varepsilon_n} \sum_{k=0}^{2m+1} R_n^k Q_n^k(a).$$

Define the map $\tilde{R}_k \colon \widetilde{C}_k \to A_\omega$ to be the one induced by

$$(\widetilde{R}_1^k, \widetilde{R}_2^k, \dots) \colon \prod_n C_{\Sigma_n^k} \to \prod_n A.$$

By Proposition 4.6 (ii), the range of R_k belongs to $A_\infty \cap B'$. Furthermore, with $e = (e_n^2)_n \in A \cap B'$, we have

$$a = ea = \sum_{k=0}^{2m+1} \widetilde{R}_k Q_k(a),$$

so that $a = \sum_{k=0}^{2m+1} \widetilde{R}_k Q_k(a)$ modulo $B^\perp$, for all $a \in A \cap B'$. □

5. Comparison in $\mathrm{F}(B, A)$

Here, we apply Theorem 4.1 to gain an understanding of Cuntz comparison in a central sequence algebra $A_\omega \cap B'$: specifically, when B has finite nuclear dimension, we are able to deduce Cuntz comparison in $A_\omega \cap B'$ from appropriate Cuntz comparisons in A_ω (at a cost which scales with the nuclear dimension of B). This allows us to prove that $\mathrm{F}(B, A)$ has M-comparison for some M, provided that B has finite nuclear dimension and A has m-comparison for some m. It also allows us to better understand fullness in $\mathrm{F}(A)$, when A is simple, has finite nuclear dimension, and has at most one trace.

The first two results will set up notation, allowing us to state the main result, Proposition 5.3. The proof of Proposition 5.3 uses the full strength of Theorem 4.1, in the sense that the specific form of the maps Q_k in the factorization is used.

Lemma 5.1. *Let a, c be two commuting positive contractions, and let $\lambda > 0$. Then*

$$[(a-\lambda)_+(c-\lambda)_+] \leqslant [(ac-\lambda)_+] \leqslant [(a-\lambda^{1/2})_+(c-\lambda^{1/2})_+].$$

Proof. The C*-algebra $C^*(a, c)$ is commutative, and hence isomorphic to $C_0(X)$ for some X. Since for $f, g \in C_0(X)_+$, we have $[f] \leqslant [g]$ iff $\forall x \in X, f(x) > 0 \Rightarrow g(x) > 0$, it suffices to prove the lemma assuming that a and c are scalars. This is not difficult. □

Lemma 5.2. *Let A be a* C*-*algebra and let B be a* C*-*subalgebra of A. Let $a, b \in A \cap B'$ be positive elements. Consider the following relations between a and b.*

(i) For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$[(ac-\varepsilon)_+] \leqslant [(bc-\delta)_+]$$

in $\mathrm{Cu}(A)$, *for all positive contractions $c \in B_+$.*

(ii) For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$[(a-\varepsilon)_+(c-\varepsilon)_+] \leqslant [(b-\delta)_+(c-\delta)_+]$$

in $\mathrm{Cu}(A)$, for all positive contractions $c \in B_+$.

(iii) $[a] \leqslant [b]$ in $\mathrm{Cu}(A \cap B')$.

Then (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii).

We shall write $a \preceq_B b$ if the equivalent conditions (i) and (ii) hold. In this case we say that a is Cuntz smaller than b by cutdowns of elements from B.

Remark. If $a = 1$ then (i) holds so long as it holds for one single $\varepsilon > 0$. Certainly, suppose that $\varepsilon_0, \delta_0 > 0$ are such that, for any $c \in B_+$, $[(c-\varepsilon_0)_+] \leqslant [(bc-\delta_0)_+]$. Given any other $\varepsilon > 0$, set $\eta := \frac{2\varepsilon}{\varepsilon_0+1}$, so that $[(c-\varepsilon)_+] = [(g_\eta(c)-\varepsilon_0)_+]$ and $[(bg_\eta(c)-\delta_0)] \leqslant [(bc-\frac{\eta\delta_0(\delta_0+1)}{2})_+]$ (proven in the same way as Lemma 5.1), so that the condition in (i) holds with $\delta := \frac{\eta\delta_0(\delta_0+1)}{2}$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is immediate from Lemma 5.1.

(iii) $\Rightarrow$ (ii): Suppose that $[a] \leqslant [b]$ in $\mathrm{Cu}(A \cap B')$. Then given $\varepsilon > 0$, there exists $\delta > 0$ and $x \in A \cap B'$ such that $(a-\varepsilon)_+ = x(b-\delta)_+x^*$; we may assume that $\delta < \varepsilon$. Thus, for $c \in B_+$,

$$(a-\varepsilon)_+(c-\varepsilon)_+ = x(b-\delta)_+x^*(c-\varepsilon)_+ = x(b-\delta)_+(c-\varepsilon)_+x^*,$$

whence $[(a-\varepsilon)_+(c-\varepsilon)_+] \leqslant [(b-\delta)_+(c-\varepsilon)_+] \leqslant [(b-\delta)_+(c-\delta)_+]$. □

Here is the main result of this section, which shows that if A is an ultraproduct algebra and B has finite nuclear dimension, then condition (i) of Lemma 5.2 implies a weakened version of (iii).

Proposition 5.3. *Let $B \subseteq A$ be C*-algebras, with B separable of nuclear dimension m, and A an ultraproduct algebra. Let $a, b_k \in A \cap B'$, with $k = 0, 1, \ldots, 2m+1$ be positive elements such that $a \preceq_B b_k$ for all k. Then*

$$[a] \leqslant \sum_{k=0}^{2m+1} [b_k]$$

in $\mathrm{Cu}(A \cap B')$.

In particular, for $a, b \in A \cap B'$, $[a] \leqslant N[b]$ in $\mathrm{Cu}(A \cap B')$ for some $N \in \mathbb{N}$ if and only if $a \preceq_B 1_M \otimes b$ for some $M \in \mathbb{N}$.

Proof. By possibly adjoining a unit to A and adding the unit of A to B, we may assume that B is a unital C*-subalgebra of A. Let $Q_k, \mathbf{C}_k, \widetilde{R}_k$ be as given by Theorem 4.1.

Given $\varepsilon > 0$, by hypothesis, there exists $\delta > 0$ such that $[(ac-\varepsilon)_+] \leqslant [(bc-\delta)_+]$ in $\mathrm{Cu}(A)$. It follows that for each positive contraction $c \in B_+$, there exists $x_{k,c} \in A$ such that

$$(ac-2\varepsilon)_+ = x_{k,c}^* x_{k,c}$$

and

$$g_\delta(b_k c)x_{k,c} = x_{k,c}.$$

In particular, $\|x_{k,c}\| \leqslant 1$ and $x_{k,c} \in \mathrm{her}(c)$.

Using the form of Q_k, it follows that there exists $y_k \in C_k$ such that

$$(Q_k(a)-2\varepsilon)_+ = y_k^* y_k$$

and

$$g_\delta(Q_k(b_k))y_k = y_k.$$

(Namely, we let $y_k = (y_{k,n})_{n=1}^\infty$ where $y_{k,n} = (x_{k,c})_{c\in\Sigma_n^k}$.) Since ε is arbitrary, we find that

$$[Q_k(a)] \leqslant [Q_k(b_k)]$$

in $\mathrm{Cu}(C_k)$. Therefore, we may find a separable subalgebra C_k' of C_k containing $Q_k(a), Q_k(b_k)$, and such that

$$[Q_k(a)] \leqslant [Q_k(b_k)] \tag{5.1}$$

in $\mathrm{Cu}(C_k')$.

Using $D = \{a, b_0, \ldots, b_{2m+1}\}$, obtain maps $\hat{R}_k \colon C_k \to (A_\infty \cap B')/B^\perp$ as in Remark 4.2. By (5.1), and since $\hat{R}_k|_{C_k'}$ is order zero, $[\hat{R}_k Q_k(a)] \leqslant [\hat{R}_k Q_k(b_k)]$.

Thus, we have

$$a = \sum_{k=0}^{2m+1} \hat{R}_k Q_k(a) \preceq \bigoplus_{k=0}^{2m+1} \hat{R}_k Q_k(b_k) \leqslant \bigoplus_{k=0}^{2m+1} b_k.$$

□

We now derive some consequences of Proposition 5.3.

Proposition 5.4. *Suppose that A has M-comparison and $B \subseteq A_\infty$ is a separable* C*-*subalgebra of nuclear dimension at most m. Then $\mathrm{F}(B, A)$ has $(2(M + 1)(m + 1) - 1)$-comparison.*

Proof. Let us suppose that $(k + 1)[a] \leqslant k[b_i]$ in the Cuntz semigroup of $\mathrm{F}(B, A)$, with $i = 1, 2, \ldots, 2(M + 1)(m + 1)$ and for some $k \in \mathbb{N}$. By Lemma 5.2 $(k + 1)[a] \leqslant_B k[b_i]$ in $\mathrm{Cu}(A_\omega)$, for all i. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that for each positive contraction $c \in B_+$, we have $(k + 1)[(ac - \varepsilon)_+] \leqslant_B k[(b_i - \delta)_+]$ in $\mathrm{Cu}(A_\omega)$. By Proposition 2.2, the C*-algebra A_ω has M-comparison, so that for each $1 \leqslant i \leqslant 2(M + 1)(m + 1) - M$, we get $[(ac - \varepsilon)_+] \leqslant [\sum_{j=i}^{i+M} (b_j c - \delta)_+]$. This, combined with Proposition 5.3, implies that $[a] \leqslant \sum_{i=1}^{2(M+1)(m+1)} [b_i]$, as desired. □

In the remainder of this section, we explore some easy consequences of Proposition 5.3 to fullness in $\mathrm{F}(A)$ for simple unital C*-algebras A, particularly those with unique trace. These consequences will not be used in the sequel. In ongoing work, the authors are further pursuing the problem of determining when an element of $\mathrm{F}(A)$ is full.

Lemma 5.5. *Let A be a unital* C*-*algebra with finite nuclear dimension. The following are equivalent:*

(i) For all $a \in \mathrm{F}(A)$, a is full in A_ω if and only if it is full in $\mathrm{F}(A)$;
(ii) For all $a \in \mathrm{F}(A)_+$, if a is full in A_ω then there exists $\gamma_a > 0$ such that

$$\tau(ac) \geqslant \gamma_a \tau(c)$$

for all $c \in A_+$ and $\tau \in \mathrm{QT}(A_\omega)$.

Proof. (i) ⇒ (ii): Suppose that $a \in \mathrm{F}(A)_+$ is full in A_ω. Then by (i) it is full in $\mathrm{F}(A)$, and so there exist $x_1, \ldots, x_k \in \mathrm{F}(A)$ such that $1 = \sum_{i=1}^k x_i a x_i^*$. Hence, for each $c \in A_+$ and $\tau \in \mathrm{QT}(A_\omega)$,

$$\tau(c) = \sum_{i=1}^k \tau(x_i a c x_i^*) \leqslant \sum_{i=1}^k \|x_i\|^2 \tau(ac),$$

and therefore (ii) holds upon setting $\gamma_a = (\sum_{i=1}^k \|x_i\|^2)^{-1}$.

(ii) $\Rightarrow$ (i): Suppose that $a \in \mathrm{F}(A)$ is full in A_ω, and let us show that it is full in $\mathrm{F}(A)$. Without loss of generality, let us assume that $a \geqslant 0$. Let $\eta > 0$ be such that $g_\eta(a)$ is still full in A_ω, and then let $K \in \mathbb{N}$ be such that $K > \gamma_{g_\eta(a)}^{-1}$. Let m denote the nuclear dimension of A. We shall show that $1 \precsim_A a \otimes 1_{(m+1)(K+1)}$, from which it follows by Proposition 5.3 that a is full.

Certainly, for $c \in A_+$ and $\tau \in \mathrm{QT}(A_\omega)$, we have

$$\begin{aligned} d_\tau((c-\eta)_+) &\leqslant \tau(g_\eta(c)) \\ &\leqslant K\tau(g_\eta(a)g_\eta(c)) \\ &\leqslant K\tau(g_{\frac{\eta^2}{4}}(ac)) \\ &\leqslant K d_\tau((ac - \frac{\eta^2}{8})_+), \end{aligned}$$

which implies that $[(c-\eta)_+] <_s (K+1)[(ac - \frac{\eta^2}{8})_+]$. By Proposition 2.2, it follows that $[(c-\eta)_+] \leqslant (m+1)(K+1)[(ac-\frac{\eta^2}{8})_+]$ in $\mathrm{Cu}(A_\omega)$. Thus, by the remark following Lemma 5.2, $1 \precsim_A a \otimes 1_{(m+1)(K+1)}$, as required. □

Theorem 5.6. *Let A be a simple unital separable* C*-*algebra with finite nuclear dimension and a unique tracial state. Then for $a \in \mathrm{F}(A)$, a is full in A_ω if and only if it is full in* $\mathrm{F}(A)$.

Remark. By [13, Theorem 1.1], if A is unital, simple, separable, nuclear, and quasidiagonal, and has a unique tracial state, then it automatically has finite nuclear dimension, so this theorem applies.

Proof. We shall use μ to denote both the unique tracial state on A and its extension to A_ω (given by taking its limit). It suffices to assume that $a \in \mathrm{F}(A)$ is positive. We shall verify that Condition (ii) of Lemma 5.5 holds with

$$\gamma_a = \inf\{\tau(a) \mid \tau \in \mathrm{QT}(A_\omega),\, \tau(1) = 1\}$$

(which is positive by the fullness of a in A_ω).

Let $\tau \in \mathrm{QT}(A_\omega)$. Since a is central, $\sigma(c) := \tau(ac)$, with $c \in A$, defines a quasitrace $\sigma\colon A \to \mathbb{C}$. Since A is exact and has a unique trace (up to a scalar multiple), we find that $\sigma = \sigma(1)\cdot\mu(\cdot)$. Plugging $c \in A_+$ on both sides and using that $\sigma(1) = \tau(a)$ we get

$$\tau(ac) = \tau(a)\mu(c).$$

If $\tau(1) = \infty$ then $\tau(a) = \infty$ (since a is full in A_ω). So $\tau(ac) = \tau(a)\mu(c)$ clearly implies that $\tau(ac) \geqslant \gamma_a \tau(c)$. Otherwise, assume that $\tau(1) = 1$. Then the restriction of τ to A agrees with μ, and so $\tau(ac) = \tau(a)\mu(c) = \tau(a)\tau(c) \geqslant \gamma_a\tau(c)$, as required. □

6. Divisibility up to cancellation in $\mathrm{F}(B, A)$

In this section, we establish the following.

Theorem 6.1. *Let A be a* C*-*algebra and let $B \subset A_\omega$ be a separable* C*-*subalgebra. Suppose that A is N-divisible for some $N \in \mathbb{N}$, and that* $\dim_{\mathrm{nuc}} B < \infty$. *Then for any $k \in \mathbb{N}$ and any $\varepsilon > 0$, there exists a c.p.c. order zero map $\phi\colon M_k(\mathbb{C}) \to \mathrm{F}(B, A)$ such that*

$$d_\tau(1 - \phi(1_k)) \leqslant \varepsilon d_\tau(1) \tag{6.1}$$

for every quasitrace $\tau \in \mathrm{QT}(\mathrm{F}(B, A))$ and

$$d_\tau(\phi(e_{11})) \geqslant \Big(\frac{1}{k} - \varepsilon\Big) d_\tau(1) \tag{6.2}$$

for every bounded $\tau \in \mathrm{QT}(\mathrm{F}(B,A))$.

Note that, by the following lemma applied to $\mathrm{F}(B,A)$, (6.2) can be reformulated as saying that

$$L[1] + p[1] \leqslant L[1] + q[\phi(e_{11})],$$

for some $L, p, q \in \mathbb{N}$, where $\frac{p}{q}$ can be taken to be arbitrarily close to $\frac{1}{k}$. Thus, in the presence of appropriate cancellation properties, it would follow that $\phi(e_{11})$ is (controllably) full, entailing divisibility of the unit of $\mathrm{F}(B,A)$.

Lemma 6.2. *Let A be a unital* C^**-algebra, let $[a] \in \mathrm{Cu}(A)$, and $\gamma > 0$. Then $d_\tau(1) \leqslant \gamma' d_\tau(a)$ for all bounded quasitraces $\tau \in \mathrm{QT}(A)$ and some $0 < \gamma' < \gamma$ if and only if $L[1] + p[1] \leqslant L[1] + q[a]$ for some $L, p, q \in \mathbb{N}$ with $\frac{q}{p} < \gamma$.*

Proof. The reverse direction is an easy computation. For the forward direction, suppose that $d_\tau(1) \leqslant \gamma' d_\tau(a)$ for all bounded $\tau \in \mathrm{QT}(A)$. Let us choose $p_0, q_0 \in \mathbb{N}$ such that $\gamma' < \frac{q_0}{p_0} < \gamma$. Then $d_\tau(1) < \frac{q_0}{p_0} d_\tau(a)$ for all bounded $\tau \in \mathrm{QT}(A)$ such that $0 < d_\tau(a) < \infty$. It follows that $d_\tau((p_0+q_0)[1]) < d_\tau(q_0([1]+[a]))$ for each $\tau \in \mathrm{QT}(A)$ such that $0 < d_\tau([1]+[a]) < \infty$; note that also $(p_0+q_0)[1] \propto q_0([1]+[a])$. Thus, by [15, Proposition 2.1] (essentially [6, Lemma 4.1]), it follows that, for some $k \in \mathbb{N}$, $k(p_0+q_0)[1] \leqslant kq_0([1]+[a])$. Now, set $L := kq_0$, $p := kp_0$ and $q := kq_0$. ☐

The proof of Theorem 6.1 is broken into two steps. First, in Lemma 6.7, we establish a much weaker form of the conclusion of Theorem 6.1, where $\phi(1_k)$ is full up to cancellation, but the degree of fullness does depend on N and $\dim_{\mathrm{nuc}} B$. Then, we use a technique to minimize the defect, removing this dependence.

Proposition 6.3. *Let A be N-almost divisible and let $B \subseteq A_\omega$ be separable and of nuclear dimension at most m. Let $d_0, \dots, d_{2m+1} \in \mathrm{F}(B,A)_+$, $k \in \mathbb{N}$ and $\varepsilon > 0$. Suppose that there exist $[a_1], \dots, [a_R], [b_1], \dots, [b_R] \in \mathrm{Cu}(A)$ and $K_1, \dots, K_R \in \mathbb{N}$ such that*

$$[a_j] \leqslant K_j[d_i] + [b_j]$$

for all $i = 0, \dots, 2m+1$ and $j = 1, \dots, R$.

Then there exist c.p.c. order zero maps $\phi_0, \dots, \phi_{2m+1} \colon M_k(\mathbb{C}) \to \mathrm{F}(B,A)$ such that

(i) $\phi_i(M_k(\mathbb{C})) \subseteq \mathrm{her}(d_i)$ for each i; and

(ii) for each j,

$$[(a_j - \varepsilon)_+] \leqslant K_j(N+1)(k+1) \sum_{i=0}^{2m+1} [\phi_i(e_{11})] + (2m+2)[b_j].$$

Proof. Let us apply Theorem 4.1, with A_ω in place of A, to obtain C^*-algebras C_i, and maps Q_i and R_i, with $i = 0, \dots, 2m+1$, as in the statement of that theorem. For each i, Q_i is an order zero map. Thus, $[Q_i(a_j)] \leqslant K_j[Q_i(d_i)] + [Q_i(b_j)]$ for each $j = 1, \dots, R$. Let $\delta > 0$ be such that, for all i, j,

$$[(Q_i(a_j) - \frac{\varepsilon}{2m+1})_+] \leqslant K_j[(Q_i(d_i) - \delta)_+] + [Q_i(b_j)].$$

Since the C^*-algebra C_i is N-almost divisible, there exists $\psi_i \colon M_k(\mathbb{C}) \to \mathrm{her}(Q_i(d_i))$ of order zero and such that $[(Q_i(d_i) - \delta)_+] \leqslant (N+1)(k+1)[\psi_i(e_{11})]$. It follows that

$$[(Q_i(a_j) - \frac{\varepsilon}{2m+1})_+] \leqslant K_j(N+1)(k+1)[\psi_i(e_{11})] + [Q_i(b_j)] \tag{6.3}$$

for each j.

Let C_i' be a unital separable C*-subalgebra of C_i which contains $Q_i(d_i)$, $Q_i(a_j)$ and $Q_i(b_j)$ for all j, and all of $\psi_i(M_k(\mathbb{C}))$. Notice that $\psi_i(M_k(\mathbb{C})) \in \overline{Q_i(d_i)C_i'Q_i(d_i)}$. Furthermore, we may enlarge C_i' if necessary – while retaining its separability – so that (6.3) holds in $\mathrm{Cu}(C_i')$. Let us use Remark 4.2, with C_i' as just described and $D := C^*(\{1\}\cup\{d_i\}\cup\{a_j, b_j\}) \subseteq \mathrm{F}(B, A)$, to obtain $\hat{R}_i\colon C_i \to \mathrm{F}(B, A)$ such that $\hat{R}_i|_{C_i'}$ is a c.p.c. map of order zero and $a = \sum_{i=0}^{2m+1} \hat{R}_i Q_i(a)$ for all $a \in D$.

For each i, let us set $\phi_i := \hat{R}_i \circ \psi_i\colon M_k(\mathbb{C}) \to \mathrm{F}(B, A)$. Let us show that these are c.p.c. order zero maps with the desired properties. Using the positivity of ϕ_i, we find that

$$\phi_i(M_k(\mathbb{C})) = \hat{R}_i(\psi_i(M_k(\mathbb{C}))) \subseteq \mathrm{her}(\hat{R}_i(Q_i(d_i))) \subseteq \mathrm{her}(d_i).$$

We note also that, for each j, since $a_j = \sum_{i=0}^{2m+1} \hat{R}_i Q_i(a_j) \approx_\varepsilon \sum \hat{R}_i((Q_i(a_j) - \frac{\varepsilon}{2m+1})_+)$,

$$\begin{aligned}
[(a_j - \varepsilon)_+] &\leqslant \sum_{i=0}^{2m+1} [\hat{R}_i((Q_i(a_j) - \frac{\varepsilon}{2m+1})_+)] \\
&\leqslant \sum_{i=0}^{2m+1} ((N+1)(k+1)[\hat{R}_i(\psi_i(e_{11}))] + [\hat{R}_i(Q_i(b_j))] \\
&= K_j(N+1)(k+1) \sum_{i=0}^{2m+1} [\phi_i(e_{11})] + (2m+2)[b_j].
\end{aligned}$$

□

Lemma 6.4. *Let A be N-almost divisible and let $B \subseteq A_\omega$ be separable and of nuclear dimension at most m. Then there exist orthogonal positive elements $d_0, d_1 \in \mathrm{F}(B, A)_+$ such that*

$$d_\tau(1) \leqslant 4(m+1)(m+2)(N+1)d_\tau(d_i), \tag{6.4}$$

for all bounded quasitraces τ on $\mathrm{F}(B, A)$ and for $i = 0, 1$.

Proof. Using $d_0 = \cdots = d_{2m+1} := 1$ which satisfy $[1] \leqslant [d_i]$, and $k = 2m+3$ in Proposition 6.3, we obtain $\phi_0, \ldots, \phi_{2m+1}\colon M_k(\mathbb{C}) \to \mathrm{F}(B, A)$ for which

$$[1] \leqslant (N+1)(2m+4) \sum_{i=0}^{2m+1} [\phi_i(e_{11})].$$

Set $[a] := \sum_{i=0}^{2m+1} [\phi_i(e_{11})]$, so that

$$[1] \leqslant (2m+2)(N+1)(2m+4)[a] \quad \text{and} \quad (2m+3)[a] \leqslant (2m+2)[1].$$

Let $\varepsilon > 0$ be such that $[1] \leqslant 4(m+1)(m+2)(N+1)[(a-\varepsilon)_+]$. Now let us define

$$\begin{aligned}
d_0 &:= g_\varepsilon(a), \\
d_1 &:= 1 - g_{\frac{\varepsilon}{2}}(a).
\end{aligned}$$

Then d_0 and d_1 are orthogonal and $[1] \leqslant 4(m+1)(m+2)(N+1)[d_0]$ in the Cuntz semigroup of $\mathrm{F}(B, A)$. Note that

$$[1] \leqslant [d_1] + [g_{\frac{\varepsilon}{2}}(a)] \leqslant [d_1] + [a].$$

Hence, by multiplying by $(2m+3)$, we get

$$\begin{aligned}
(2m+3)[1] &\leqslant (2m+3)[d_1] + (2m+3)[a] \\
&\leqslant (2m+3)[d_1] + (2m+2)[1].
\end{aligned}$$

Applying d_τ, where τ is a bounded quasitrace, and then cancelling yields (6.4). □

Lemma 6.5. *Let A be a unital C*-algebra and let $b, c \in A_+$ be positive commuting elements. Let $\gamma > 0$. If $d_\tau(1) \leqslant \gamma d_\tau(c)$ for every (bounded) $\tau \in \mathrm{QT}(A \cap \{b\}')$ then $d_\tau(b) \leqslant \gamma d_\tau(bc)$ for every (bounded) $\tau \in \mathrm{QT}(A)$.*

Proof. For each (bounded) quasitrace τ on A, define $\hat{\tau}\colon (A \cap \{b\}')_+ \to [0, \infty]$ by $\hat{\tau}(x) := \sup \tau(b^{1/n}x)$. It is easy to see that $\hat{\tau}$ is a (bounded) quasitrace, and so

$$d_\tau(b) = d_{\hat{\tau}}(1) \leqslant \gamma d_{\hat{\tau}}(c) = \gamma d_\tau(bc),$$

as required. □

Proposition 6.6. *Given $N, m \in \mathbb{N}$, there exists $P(N, m, i) \in \mathbb{N}$ for $i = 0, 1, \ldots$ such that the following holds: If A is N-almost divisible and $B \subseteq A_\omega$ is separable and has nuclear dimension at most m, then there exist pairwise orthogonal positive elements $d_0, d_1, \dots \in \mathrm{F}(B, A)$ such that*

$$d_\tau(1) \leqslant P(N, m, i) d_\tau(d_i)$$

for each i and each bounded quasitrace τ on $\mathrm{F}(B, A)$.

Proof. This follows easily using Lemmas 6.4 and 6.5: We begin by getting two positive orthogonal elements $d_0^0, d_1^0 \in \mathrm{F}(B, A)$ satisfying (6.4). We note that $\mathrm{F}(B, A) \cap \{d_0^0, d_1^0\} \cong \mathrm{F}(B', A)$ where $B' := \mathrm{C}^*(B \cup \{d_0^0, d_1^0\})$, and by [27, Lemma 7.1], $\dim_{\mathrm{nuc}} B' \leqslant 2m - 1$. Thus by Lemma 6.4, we get two more positive orthogonal elements $d_0^1, d_1^1 \in \mathrm{F}(B, A) \cap \{d_0^0, d_1^0\}'$ satisfying (6.4) but with $2m - 1$ in place of m. Hence, $d_0 := d_0^0$, $d_1 := d_1^0 d_0^1$, and $d_1^0 d_1^1$ are positive orthogonal elements in $\mathrm{F}(B, A)$. Using Lemma 6.5, we get

$$d_\tau(1) \leqslant 4(m+1)(m+2)(N+1) \cdot 4(m+2)(m+3)(N+1) d_\tau(d_1^0 d_i^1)$$

for $i = 0, 1$. The entire sequence $(d_i)_{i=1}^\infty$ is obtained by continuing in this manner, and we find that

$$P(N, m, i) := 4(m+1)(m+2)(N+1)\Big(4(m+2)(m+3)(N+1)\Big)^i$$

works. □

Lemma 6.7. *Given $N, m \in \mathbb{N}$, there exists $Q(N, m)$ such that the following holds: If A is N-almost divisible and $B \subset A_\omega$ is separable and has nuclear dimension at most m, then for each $k \in \mathbb{N}$ there exists a c.p.c. order zero map $\phi\colon M_k(\mathbb{C}) \to \mathrm{F}(B, A)$ such that*

$$d_\tau(1) \leqslant Q(N, m) d_\tau(\phi(1_k)) \tag{6.5}$$

for all bounded quasitraces τ on $\mathrm{F}(B, A)$.

Proof. Using the constants from Proposition 6.6, set

$$P := \max\{P(N, m, i) \mid i = 0, \ldots, 2m+1\}$$

and $Q(N, m) := 8P \cdot (N + 1)$.

Given A and B as in the statement, let us use Proposition 6.6 to get orthogonal positive elements $d_0, \ldots, d_{2m+1} \in \mathrm{F}(B, A)_+$ such that $d_\tau(1) \leqslant P d_\tau(d_i)$ for each i and each bounded quasitrace τ on $\mathrm{F}(B, A)$. By Lemma 6.2, for each i there exist $L_i, p_i, q_i \in \mathbb{N}$ such that $\frac{p_i}{q_i} > \frac{1}{2P}$ and

$$L_i[1] + p_i[1] \leqslant L_i[1] + q_i[d_i].$$

Setting $L := \max_i L_i$, it follows that $L[1] + p_i[1] \leqslant L[1] + q_i[d_i]$ for all i. Furthermore, with $p := \prod_{i=0}^{2m+1} p_i$ and $q := p \cdot \max_i \frac{q_i}{p_i}$, we have $\frac{p}{q} > \frac{1}{2P}$ and $L[1] + p[1] \leqslant L[1] + q[d_i]$ for all i. From this, we obtain that

$$L[1] + np[1] \leqslant L[1] + nq[d_i] \tag{6.6}$$

for all i and all $n = 1, 2, \ldots$. Let us fix n large enough (how large value will be specified soon). Feeding $d_0, \ldots, d_{2m+1}$ and (6.6) to Proposition 6.3, we obtain c.p.c. order zero maps $\phi_0, \ldots, \phi_{2m+1} \colon M_k(\mathbb{C}) \to \mathrm{F}(B, A)$ such that $\phi_i(M_k(\mathbb{C})) \subseteq \mathrm{her}(d_i)$ and

$$L[1] + np[1] \leqslant (2m+2)L[1] + 2nq(N+1) \sum_{i=0}^{2m+1} [\phi_i(1_k)]. \tag{6.7}$$

Since the d_i's are orthogonal, it follows that $\phi := \sum_{i=0}^{2m+1} \phi_i$ is a c.p.c. order zero map. Moreover, (6.7) implies that for each bounded quasitrace $\tau \in \mathrm{QT}(\mathrm{F}(B, A))$ we have

$$\frac{L + np}{nq} d_\tau(1) \leqslant \frac{(2m+2)L}{nq} d_\tau(1) + 2(N+1) d_\tau(\phi(1)).$$

Observe that $\frac{1}{2P} < \frac{L+np}{nq}$ for all n while $\frac{(2m+2)L}{nq} \to 0$ as $n \to \infty$. It is now clear that choosing n large enough we will have $\frac{1}{4P} d_\tau(1) \leqslant 2(N+1) d_\tau(\phi(1))$ for every bounded quasitrace τ, as required. □

In the remainder of this section, we show how to get from the conclusion of Lemma 6.7 to the conclusion of Theorem 6.1. This step can be stated in a very general form, as follows.

Proposition 6.8. *Let A be a C*-algebra, let $B \subset A_\omega$ be a separable C*-subalgebra, and let $k \in \mathbb{N}$. Suppose that there exists $Q > 0$ such that, for every c.p.c. order zero map $\phi \colon M_k(\mathbb{C}) \to A_\omega \cap B'$ and $C := C^*(B \cup \phi(M_k(\mathbb{C})))$, there exists a c.p.c. order zero map $\psi \colon M_k(\mathbb{C}) \to \mathrm{F}(C, A)$ such that*

$$d_\tau(1) \leqslant Q d_\tau(\psi(1)). \tag{6.8}$$

for every bounded quasitrace τ on $\mathrm{F}(C, A)$, Then, for every $\varepsilon > 0$, there exists a c.p.c. order zero map $\psi \colon M_k(\mathbb{C}) \to \mathrm{F}(B, A)$ such that

$$[1 - \psi(1)] <_s \varepsilon[1]. \tag{6.9}$$

Prior to proving this proposition, let us see how to prove Theorem 6.1 using it and Lemma 6.7.

Proof of Theorem 6.1. Let A, B be as in Theorem 6.1, and set $m := \dim_{\mathrm{nuc}} B$. We first show that we can find ϕ for which (6.1) holds, then show that (6.2) follows. For this, we wish to apply Proposition 6.8, with $Q := Q(N, 2m-1)$ as given by Lemma 6.7.

For a C*-algebra C as in the statement of Proposition 6.8, we have by [27, Lemma 7.1] that $\dim_{\mathrm{nuc}} C \leqslant 2m - 1$. Thus, Lemma 6.7 tells us that there exists $\psi \colon M_k(\mathbb{C}) \to \mathrm{F}(C, A)$ such that $d_\tau(1) \leqslant Q[\psi(1)]$, verifying the hypothesis of Proposition 6.8, and therefore ϕ exists satisfying (6.1).

Now, given that ϕ satisfies (6.1), we have $[1] \leqslant [\phi(1)] + [(1 - \phi(1)]$, and therefore, for any $\tau \in \mathrm{QT}(\mathrm{F}(B, A))$,

$$d_\tau(1) \leqslant d_\tau(\phi(1)) + d_\tau(1 - \phi(1)) \leqslant d_\tau(\phi(1)) + \varepsilon d_\tau(1).$$

When τ is bounded, we may cancel to get $d_\tau(\phi(1)) \geqslant (1 - \varepsilon) d_\tau(1)$, so that

$$d_\tau(\phi(e_{11})) \geqslant \frac{1 - \varepsilon}{k} d_\tau(1) \geqslant (\frac{1}{k} - \varepsilon) d_\tau(1).$$ □

Some preparation is needed before we prove Proposition 6.8. First, we will need the following result by Winter:

Lemma 6.9. *Let D be a C*-algebra and $\phi_1, \phi_2 \colon M_k(\mathbb{C}) \to D$ be c.p.c. order zero maps with ranges that commute.*

(i) If $\phi_1(1)+\phi_2(1)\leqslant 1$ *then there exists a c.p.c. order zero map* $\phi\colon M_k(\mathbb{C})\to D$ *such that* $\phi(1)=\phi_1(1)+\phi_2(1)$.

(ii) There exists a c.p.c. order zero map $\phi\colon M_k(\mathbb{C})\to D$ *such that* $\phi(1)\geqslant\phi_1(1)$ *and*

$$(1-\phi(1))=(1-\phi_1(1))(1-\phi_2(1)).$$

Proof. (i) This is [36, Lemma 2.3] (cf. also [10, Lemma 7.6]).

(ii) This follows from (i) applied to ϕ_1 and $(1-\phi_1(1))\phi_2$. □

We record a small functional calculus maneouvre in the following lemma, that allows us to strengthen (6.8).

Lemma 6.10. *Let* A *be a unital* C*-*algebra, let* $\phi\colon M_k(\mathbb{C})\to A$ *be a c.p.c. order zero map. If* $d_\tau(1)\leqslant Q' d_\tau(\phi(1))$ *for all bounded quasitraces* τ *on* A*, for some* $Q>0$*, then there exists a c.p.c. order zero map* $\psi\colon M_k(\mathbb{C})\to A$ *such that*

$$d_\tau(1-\psi(1))\leqslant(1-\frac{1}{2Q})d_\tau(1). \tag{6.10}$$

Proof. Using Lemma 6.2, we can see that there exists $\delta>0$ such that $d_\tau(1)\leqslant 2Q\cdot d_\tau((\phi(1)-\delta)_+)$ for all bounded quasitraces τ. Let us set $\psi=g_{\frac{\delta}{2}}(\phi)$. If τ is unbounded, (6.10) holds automatically. Otherwise,

$$\begin{aligned}2Q\cdot d_\tau(1-\psi(1))+d_\tau(1)&\leqslant 2Q\cdot d_\tau(1-\psi(1))+2Q\cdot d_\tau((\phi-\delta)_+)\\&\leqslant 2Q\cdot d_\tau(1),\end{aligned}$$

and from here we can cancel $d_\tau(1)$ to get (6.10). □

Proof of Proposition 6.8. Let us set β to be the infimum of $\varepsilon>0$ for which there exists $\phi\colon M_k(\mathbb{C})\to \mathrm{F}(B,A)$ such that $[1-\phi(1)]<_s\varepsilon\cdot[1]$. It is clear by the hypothesis that $\beta\leqslant 1$. We must show that $\beta=0$, and to do this, we shall show that β satisfies

$$\beta\leqslant\Big(1-\frac{1}{3Q}\Big)\beta.$$

Let $\varepsilon>\beta$ and let $\psi\colon M_k(\mathbb{C})\to\mathrm{F}(B,A)$ be such that $d_\tau(1-\psi(1))\leqslant\varepsilon\cdot d_\tau(1)$. Let us lift ψ to a c.p.c. order zero map $\hat{\psi}\colon M_k(\mathbb{C})\to A_\omega\cap B'$ and set $C:=C^*(B\cup\hat{\psi}(M_k(\mathbb{C})))$. By the hypothesis and Lemma 6.10, there exists $\phi_0\colon M_k(\mathbb{C})\to\mathrm{F}(C,A)$ such that

$$d_\tau(1-\phi_0(1))\leqslant(1-\frac{1}{2Q})d_\tau(1) \tag{6.11}$$

for all quasitraces τ. Notice that $\mathrm{F}(C,A)\equiv\mathrm{F}(B,A)\cap\psi(M_k(\mathbb{C}))'$, so that we can view ϕ_0 and ψ as c.p.c. order zero maps into $\mathrm{F}(B,A)$ with commuting ranges.

By Lemma 6.9, there exists $\phi\colon M_k(\mathbb{C})\to\mathrm{F}(B,A)$ such that $(1-\phi(1))=(1-\psi(1))(1-\phi_0(1))$. For any bounded trace τ on $\mathrm{F}(B,A)$, using Lemma 6.5 and (6.11), we have

$$\begin{aligned}d_\tau(1-\phi(1))&=d_\tau((1-\psi(1))(1-\phi_0(1)))\\&\leqslant(1-\frac{1}{2Q})d_\tau(1-\psi(1))\\&\leqslant(1-\frac{1}{2Q})\varepsilon\cdot d_\tau(1).\end{aligned}$$

Hence, $[1-\phi(1)]<_s(1-\frac{1}{3Q})\varepsilon[1]$. This shows that $\beta\leqslant(1-\frac{1}{3Q})\varepsilon$ for any $\varepsilon>\beta$, and so $\beta\leqslant(1-\frac{1}{3Q})\beta$. □

7. $\mathcal{Z}$-stability

This section contains the proofs of conjectures (C1) and (C2) in various cases.

7.1. **The simple case.** Here, we give a simplified proof of the main results of [37] and [27]:

Theorem 7.1. *([37, Theorem 7.1], [27, Theorem 8.5]) Let A be a simple, separable, stably finite* C*-*algebra which is (M, N)-pure, for some $M, N \in \mathbb{N}$, and which has locally finite nuclear dimension. Then A is $\mathcal{Z}$-stable.*

The key step is going from the conclusion of Theorem 6.1 to a nontracial version, which requires a certain finiteness condition on $\mathrm{F}(B, A)$, namely that the unit is not stably properly infinite in any quotient. We shall see that this finiteness condition holds when A is simple and tracial, has M-comparison for some M and $B \subseteq A$ has finite nuclear dimension. Interestingly, even if A is an infinite UHF algebra, if $B \subseteq A_\omega$ (instead of $\subseteq A$) with finite nuclear dimension, $\mathrm{F}(B, A)$ may have purely infinite quotients, as shown in Example 7.6. At the end of this subsection we given a separate argument that deals with the simple purely infinite case.

Proposition 7.2. *Let A be a* C*-*algebra and let $B \subset A_\omega$ be a separable* C*-*subalgebra. Suppose that A is (M, N)-pure, that $\dim_{\mathrm{nuc}} B < \infty$, and that 1 is not stably properly infinite in any quotient of $\mathrm{F}(B, A)$. Then there exists a unital embedding of $\mathcal{Z}$ into $\mathrm{F}(B, A)$.*

Proof. We shall show that, for each $k \in \mathbb{N}$, there exists a unital *-homomorphism from $\mathcal{Z}_{k,k+1}$ to $\mathrm{F}(B, A)$. By a diagonal sequence argument this implies that $\mathcal{Z}$ embeds unitally in $\mathrm{F}(B, A)$ (see [27, Proposition 5.3] for the case of $\mathrm{F}(A)$).

Let us fix $k \in \mathbb{N}$. By Proposition 5.4, there exists $\overline{M} \in \mathbb{N}$ such that $\mathrm{F}(B, A)$ has $\overline{M}$-comparison. By Theorem 6.1, there exists a c.p.c. map $\phi\colon M_{k(\overline{M}+1)}(\mathbb{C}) \to \mathrm{F}(B, A)$ of order zero such that $d_\tau(1 - \phi(1)) < \frac{1}{2(\overline{M}+1)k+1} d_\tau(1)$ and $\frac{1}{2(\overline{M}+1)k} d_\tau(1) < d_\tau(\phi(e_{11}))$ for every bounded quasitrace τ on $\mathrm{F}(B, A)$. By Lemma 6.2, we have

$$L[1] + p[1] \leqslant L[1] + q[\phi(e_{11})]$$

for some $L, p, q \in \mathbb{N}$ with $\frac{p}{q} > \frac{1}{2(\overline{M}+1)k}$. Let $\varepsilon > 0$ be such that $L[1] + p[1] \leqslant L[1] + q[(\phi(e_{11}) - \varepsilon)_+]$. This implies that 1 is not stably properly infinite modulo the ideal generated by $(\phi(e_{11}) - \varepsilon)_+$. So by the hypothesis, $(\phi(e_{11}) - \varepsilon)_+$ is full.

For every bounded quasitrace τ on $\mathrm{F}(B, A)$ we have

$$d_\tau(1 - \phi(1)) < \frac{1}{2(\overline{M}+1)k+1} d_\tau(1) < \gamma d_\tau((\phi(e_{11}) - \varepsilon)_+)$$

for some $\gamma < 1$. On the other hand, if τ is unbounded then

$$d_\tau(1 - \phi(1)) \leqslant \infty = d_\tau((\phi(e_{11}) - \varepsilon)_+).$$

Thus, $[1 - \phi(1)] <_s [(\phi(e_{11}) - \varepsilon)_+]$, and therefore by $\overline{M}$-comparison in $\mathrm{F}(B, A)$ we have $[1 - \phi(1)] \leqslant (\overline{M} + 1)[(\phi(e_{11}) - \varepsilon)_+]$. Let us now view $M_{k(\overline{M}+1)}(\mathbb{C})$ as $M_k(\mathbb{C}) \otimes M_{\overline{M}+1}(\mathbb{C})$, and set $\psi := \phi(\,\cdot \otimes 1_{\overline{M}+1})\colon M_k(\mathbb{C}) \to \mathrm{F}(B, A)$. We can restate our latest conclusion as $[1 - \psi(1)] \leqslant [(\psi(e_{11}) - \varepsilon)_+]$. By [23, Proposition 5.1], it follows that there is a unital *-homomorphism $\mathcal{Z}_{k,k+1} \to \mathrm{F}(B, A)$, as required. □

Now, we verify the above finiteness condition. We will need some lemmas that will be reused in the sequel. Let us say that a Cuntz class $[c]$ is **pseudocompact** if $[c] \propto [(c - \varepsilon)_+]$ for some $\varepsilon > 0$. If $[c]$ is pseudocompact then $\mathrm{Ideal}(c)$ is a compact open set of $\mathrm{Prim}(A)$. Conversely, if $\mathrm{Ideal}(c)$ is compact then $[(c - t)_+]$ is pseudocompact for all sufficiently small $t > 0$.

Recall that a Cuntz class $[c]$ is said to be stably properly infinite if it is nonzero and $(n+1)[c] = n[c]$ for some $n \in \mathbb{N}$.

Lemma 7.3. *Let A be a* C^**-algebra with M-comparison and such that no nonzero simple subquotient of A is purely infinite.*

(i) Then no quotient of A contains a pseudocompact, stably properly infinite element.

(ii) If $[c]$ is pseudocompact and $L[c] + p[c] \leqslant L[c] + q[b]$ for some $[b]$ and $p, q > 0$ then $[c] \leqslant (M+1)k[b]$ for any $k > \frac{q}{p}$.

Proof. (i) It suffices to show that $\mathrm{Cu}(A)$ contains no pseudocompact, stably properly infinite element. Assume for a contradiction that $[c]$ is pseudocompact and stably properly infinite. Then a sufficiently large multiple of $[c]$ is compact and properly infinite. Let J be a maximal ideal not containing c. Then $\mathrm{Ideal}(c)/J$ is simple, has M-comparison, and a sufficiently large multiple of $[\pi_J(c)]$ is compact and properly infinite. It follows that $\mathrm{Ideal}(c)/J$ is a purely infinite C^*-algebra (see the proof of Proposition 2.3), which contradicts our hypotheses.

(ii) Let $I = \mathrm{Ideal}(b)$. Passing to the quotient by I we get $(L+p)[\pi_I(c)] = L[\pi_I(c)]$. Since $[\pi_I(c)]$ cannot be stably properly infinite, it must be 0. That is, c belongs to the ideal generated by b. For any $\tau \in \mathrm{QT}(A)$, if $d_\tau(b) < \infty$ then $d_\tau((c-t)_+) < \infty$ for all $t > 0$ and so $d_\tau(c) < \infty$ by the pseudocompactness of $[c]$. Hence, from the relation $L[c] + p[c] \leqslant L[c] + q[b]$ we get that $d_\tau(c) \leqslant \frac{q}{p} d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$ and so $[c] <_s k[b]$. Since A has M-comparison, we conclude that $[c] \leqslant (M+1)k[b]$. □

Lemma 7.4. *Suppose that A has M-comparison. If no quotient of A contains a stably properly infinite compact element then the same is true for $\prod_{i=0}^\infty A$ and A_ω.*

Proof. This property clearly passes to quotients, so we prove it just for $\prod A$. That M-comparison passes is shown in Proposition 2.2. Suppose that $[a] \in \mathrm{Cu}(\prod A)$ becomes stably properly infinite and compact in some quotient. This means that

$$(n+1)[a] \leqslant n[(a-\bar{\varepsilon})_+] + [b]$$

in $\mathrm{Cu}(\prod A)$, where a is not in the ideal generated by b. Let $\varepsilon > 0$ and find $\delta > 0$ such that

$$(n+1)[(a-\varepsilon)_+] \leqslant n[(a-\bar{\varepsilon})_+] + [(b-\delta)_+].$$

Then for each i we have

$$(n+1)[(a_i-\varepsilon)_+] \leqslant n[(a_i-\bar{\varepsilon})_+] + [(b_i-\delta)_+].$$

Let us assume without loss of generality that $\varepsilon < \bar{\varepsilon}$. Then, with $I := \mathrm{Ideal}((b_i-\delta)_+)$, we have

$$(n+1)[(\pi_I(a_i-\varepsilon)_+] \leqslant n[\pi_I((a_i-\bar{\varepsilon})_+],$$

so that by Lemma 7.3 (i) we must have $(a_i-\varepsilon)_+ \in \mathrm{Ideal}((b_i-\delta)_+)$. Arguing as in the proof of Lemma 7.3 (ii), we get that $d_\tau((a_i-\varepsilon)_+) \leqslant d_\tau((b_i-\delta)_+)$ for all quasitraces $\tau \in \mathrm{QT}(A)$ (it suffices to consider those τ for which $d_\tau((b_i-\delta)_+) < \infty$). Thus, $[(a_i-\varepsilon)_+] <_s 2[(b_i-\delta)_+]$ and by M-comparison, $[(a_i-\varepsilon)_+] \leqslant 2(M+1)[(b_i-\delta)_+]$ for all i. It follows that $[(a-\varepsilon)_+] \leqslant 2(M+1)[b]$. Since $\varepsilon > 0$ is arbitrary, we get that $[a] \leqslant 2(M+1)[b]$, and in particular a belongs to the ideal generated by b. This is a contradiction. □

Proposition 7.5. *Let A be a simple tracial* C^**-algebra with M-comparison for some M. Let $B \subseteq A$ be a* C^**-subalgebra of nuclear dimension at most m. Then in every quotient of* $\mathrm{F}(B, A)$*, 1 is not stably properly infinite.*

Proof. For a contradiction, suppose that in some nonzero quotient of $\mathrm{F}(B,A)$, we have $(k+1)[1] \leqslant k[1]$. Equivalently, there exists a non-full element $[b] \in \mathrm{Cu}(\mathrm{F}(B,A))$ such that $(k+1)[1] \leqslant k[1]+[b]$. Consequently, for some $\varepsilon > 0$, $(k+1)[1] \leqslant k[1] + [(b-\varepsilon)_+]$.

For $c \in B_+$ and $\eta > 0$, we shall show that $[(c-\eta)_+] \leqslant (M+1)[(b-\varepsilon)_+(c-\eta)_+]$ in $\mathrm{Cu}(A_\omega)$. We know that $(k+1)[(c-\eta)_+] \leqslant k[(c-\eta)_+] + [(b-\varepsilon)_+(c-\eta)_+]$. Since A is simple, $(c-\eta)_+$ is pseudocompact, so that by Lemma 7.4 and Lemma 7.3 (ii),

$$[(c-\eta)_+] \leqslant (M+1)[(b-\varepsilon)_+(c-\eta)_+],$$

as required.

It now follows by Proposition 5.3 that

$$[1] \leqslant 2(m+1)(M+1)[b],$$

which is a contradiction, since $[b]$ is not full. □

Proof of Theorem 7.1. For every $B \subseteq A$ of finite nuclear dimension, Proposition 7.2 and Proposition 7.5 combine to tell us that $\mathcal{Z}$ embeds unitally in $\mathrm{F}(B,A)$. Since A is separable and has locally finite nuclear dimension, a diagonal sequence argument implies that $\mathcal{Z}$ embeds unitally in $\mathrm{F}(A)$. It follows by Proposition 2.4 that A is $\mathcal{Z}$-stable. □

Here is an example to show that the conclusion of Proposition 7.5 fails if we allow B to be positioned in A_ω instead of in A.

Example 7.6. Let A be an infinite dimensional UHF algebra. By [8, 33], $C_0((0,1]) \otimes \mathcal{O}_2$ is quasidiagonal, and therefore there exists an embedding

$$\phi\colon C_0((0,1]) \otimes \mathcal{O}_2 \to A_\omega.$$

Set $B := \phi(C_0((0,1]) \otimes 1_{\mathcal{O}_2}) \subseteq A_\omega$. Since B is commutative, $\mathrm{F}(B,A) = A_\omega \cap B'$ contains B. By unitizing, we see that $\mathrm{F}(B,A)$ is a $C([0,1])$-algebra.

Let us see that the quotient of $\mathrm{F}(B,A)$ given by the fibre at 1 is infinite. Surely, it is clear that $\phi(C_0((0,1]) \otimes \mathcal{O}_2)) \subseteq \mathrm{F}(B,A)$. Therefore, the fibre at 1 contains a copy of $\mathcal{O}_2$, which implies that it is infinite. What is more, we may pick a simple quotient of this fibre (which is of course a quotient of $\mathrm{F}(B,A)$), and it will have 3-comparison by Proposition 5.4, which implies that it is purely infinite.

If a simple C*-algebra is traceless and has M-comparison then it is purely infinite (see the proof of Proposition 2.3). If in addition the C*-algebra is nuclear, then it is $\mathcal{O}_\infty$-stable by Kirchberg's theorem and a fortiori also $\mathcal{Z}$-stable. Below, we give an independent proof of $\mathcal{Z}$-stability for simple separable purely infinite C*-algebras with locally finite nuclear dimension.

Proposition 7.7. *Let A be a C*-algebra that is separable, unital, simple, purely infinite, and of locally finite nuclear dimension. Then A is $\mathcal{Z}$-stable.*

Proof. Since A is simple and separable, we have $\mathrm{F}(A) \neq \mathbb{C}$ by [9, Lemma 2.8]. Thus, there exist non-zero orthogonal positive elements $d^0, d^1 \in \mathrm{F}(A)$. Let us choose $0 < \delta < \|d^0\|, \|d^1\|$. Fix $i = 0,1$ and consider the set $\{c \in A \mid c(d^i - \delta)_+ = 0\}$. This is a closed two-sided ideal of A. Hence, it is either $\{0\}$ or A. It cannot be the latter, since 1 is not in it. Thus, $c(d^0-\delta)_+ \neq 0$ for all non-zero $c \in A$. Since A_ω is simple and purely infinite, $c \precsim c(d^i-\delta)_+$ in A_ω, for all positive $c \in A$. Applied to $(c-\varepsilon)_+$ for a fixed $\varepsilon > 0$ we get $(c-\varepsilon)_+ \precsim (c-\varepsilon)_+(d^i-\delta)_+$. Thus, $1 \precsim_A d^i$ for $i = 0,1$.

Let $B \subset A_\omega$ be a separable C*-subalgebra. By a standard argument passing to subsequences applied to d^0 and d^1, we can find positive orthogonal elements $\tilde{d}^0, \tilde{d}^1 \in \mathrm{F}(B,A)$ such that $1 \precsim_B \tilde{d}^i$ for $i = 0,1$. Suppose that the nuclear dimension of B is at most m. Then by

Proposition 5.4 we have $[1] \leqslant (2m+1)[\tilde{d^i}]$ in the Cuntz semigroup of $\mathrm{F}(B,A)$ for $i=0,1$. Thus, by Theorem 7.8, A is $\mathcal{Z}$-stable. □

7.2. **Full orthogonal elements in** $\mathrm{F}(A)$.

Theorem 7.8. *Let A be a separable* C^**-algebra which is (M,N)-pure, for some $M,N \in \mathbb{N}$, and which has locally finite nuclear dimension. Suppose that for each $m \in \mathbb{N}$, there exist $P_m \in \mathbb{N}$ such that the following holds: If $B \subset A_\omega$ is a separable* C^**-subalgebra with nuclear dimension at most m, then there exist orthogonal elements $d_0, d_1 \in \mathrm{F}(B,A)_+$ such that $[1] \leqslant P_m[d_i]$ for $i=0,1$. Then A is $\mathcal{Z}$-stable.*

Remark. (i) In particular, the above result applies when A is separable C^*-algebra, (M,N)-pure, has locally finite nuclear dimension, and $\mathrm{F}(A)$ contains two orthogonal full elements.

(ii) The above theorem has a strong converse: if A is $\mathcal{Z}$-stable then it is $(0,0)$-pure, and $\mathrm{F}(B,A)_+$ has orthogonal elements d_0, d_1 which satisfy $[1] \leqslant 3[d_i]$ for $i=0,1$. Certainly, $(0,0)$-pureness is shown (essentially) in [37, Proposition 3.7] (primarily using [21]). Also, it is well-known that $\mathcal{Z}$ contains orthogonal elements $\hat{d}_0, \hat{d}_1$ such that $[1] \leqslant 3[(\hat{d}_i - \varepsilon)_+]$, for $i=0,1$, and some $\varepsilon > 0$. Viewing A as $A \otimes \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$, we can easily use these to produce orthogonal elements $d_0, d_1 \in \mathrm{F}(A)_+$ such that $[1] \leqslant 3[d_i]$. For $\mathrm{F}(B,A)$, we simply use a speeding-up argument, as in the proof of [37, Proposition 4.4].

Proof. This proof contracts ideas found in the proofs of Theorems 6.1 and 7.1. We must show that for each B of finite nuclear dimension and each $k \in \mathbb{N}$, there exists a unital *-homomorphism $\mathcal{Z}_{k,k+1} \to \mathrm{F}(B,A)$.

Using the idea behind the proof of Proposition 6.6, we see that there exists $Q_m \in \mathbb{N}$ such that, if $B \subseteq A_\omega$ is a separable C^*-subalgebra of nuclear dimension at most m, then there exist orthogonal elements $d_0, \ldots, d_{2m+1} \in \mathrm{F}(B,A)_+$ such that $[1] \leqslant Q_m[d_i]$.

Fixing $B \subseteq A_\omega$ of nuclear dimension at most $m < \infty$, let us show that the hypothesis of Proposition 6.8 holds, with $Q := Q_{2m-1} \cdot N$. Let C be a C^*-algebra as in the statement of Proposition 6.8; it has nuclear dimension at most $\overline{m} := 2m-1$. Therefore, let $d_0, \ldots, d_{2\overline{m}+1} \in \mathrm{F}(C,A)_+$ be orthogonal, such that $[1] \leqslant Q_{\overline{m}}[d_i]$. Using this with Proposition 6.3, we get c.p.c. order zero maps $\phi_0, \ldots, \phi_{2\overline{m}+1} \colon M_k(\mathbb{C}) \to \mathrm{F}(C,A)$ such that $\phi_i(M_k(\mathbb{C})) \subseteq \mathrm{her}(d_i)$ for each i and $[1] \leqslant Q_{\overline{m}} N \sum_{i=0}^{2m+1} [\phi_i(1_k)]$. Since the d_i's are pairwise orthogonal, it follows that $\phi := \sum_{i=0}^{2m+1} \phi_i$ is a c.p.c. order zero map, and we see that $[1] \leqslant Q[\phi(1)]$, as required.

By Proposition 6.8, for any $\varepsilon > 0$, we may find a c.p.c. order zero map $\phi_1 \colon M_k(\mathbb{C}) \to \mathrm{F}(B,A)$ such that $[1 - \phi_1(1)] <_s \varepsilon[1]$. By the argument above, we may then find another c.p.c. order zero map $\phi_2 \colon M_k(\mathbb{C}) \to \mathrm{F}(B,A) \cap \phi_0(M_k(\mathbb{C}))'$ such that $[1] \leqslant Q[\phi_2(1)]$. Then, we may effectively combine these two order zero maps by Lemma 6.9 (ii), to get a c.p.c. order zero map $\phi \colon M_k(\mathbb{C}) \to \mathrm{F}(B,A)$ such that $[1] \leqslant Q[\phi(1)]$ and $[1 - \phi(1)] <_s \varepsilon[1]$.

By Proposition 5.4, $\mathrm{F}(B,A)$ has $\overline{M}$-comparison for some $\overline{M} \in \mathbb{N}$. Now, given $k \in \mathbb{N}$, as explained in the previous paragraph, we may find a c.p.c. order zero map $\phi \colon M_{k(\overline{M}+1)}(\mathbb{C}) \to \mathrm{F}(B,A)$ such that $[1] \leqslant Q[\phi(1)]$ and $[1 - \phi(1)] <_s \frac{1}{k(\overline{M}+1)Q}[1]$. Let $\varepsilon > 0$ be such that $[1] \leqslant Q[(\phi(1) - \varepsilon)_+]$. Combining these, we see that, $[1 - \phi(1)] <_s [(\phi(e_{11}) - \varepsilon)_+]$, so that by $\overline{M}$-comparison, $[1 - \phi(1)] \leqslant (\overline{M}+1)[(\phi(e_{11}) - \varepsilon)_+]$.

Let us now view $M_{k(\overline{M}+1)}(\mathbb{C})$ as $M_k(\mathbb{C}) \otimes M_{\overline{M}+1}(\mathbb{C})$, and set $\psi := \phi(\,\cdot \otimes 1_{\overline{M}+1}) \colon M_k(\mathbb{C}) \to \mathrm{F}(B,A)$. We can restate our latest conclusion as $[1 - \psi(1)] \leqslant [(\psi(e_{11}) - \varepsilon)_+]$. By [23, Proposition 5.1], it follows that there is a unital *-homomorphism $\mathcal{Z}_{k,k+1} \to \mathrm{F}(B,A)$, as required. □

7.3. **C*-algebras with finite subquotients and a basis of compact-open sets for the spectrum.** Here, we show the following:

Theorem 7.9. *Let A be a separable* C**-algebra of locally finite nuclear dimension which is (M, N)-pure for some $M, N > 0$. Suppose also that*

(i) no nonzero simple subquotient of A is purely infinite; and
(ii) Prim(A) *has a basis of compact open sets.*

Then A is $\mathcal{Z}$-stable.

Combined with Theorem 3.1, the preceding theorem yields at once the following

Corollary 7.10. *Let A be a separable* C**-algebra of finite nuclear dimension and with no elementary subquotients. Suppose also that A satisfies conditions (i) and (ii) of the previous theorem. Then A is $\mathcal{Z}$-stable.*

Before getting towards the proof of Theorem 7.9, let us point out that the conditions (i) and (ii) of that theorem, together with having finite nuclear dimension, hold in the following cases:

(a) If A has finite decomposition rank and the ideal property (as defined in [22, Definition 1.5.2]). In particular, this is the case if A has finite decomposition rank and real rank zero.
(b) If $A = C(X) \rtimes_\alpha \mathbb{Z}^n$, where X is the Cantor set and $\alpha\colon \mathbb{Z}^n \to \mathrm{Aut}(A)$ is a free action. Indeed, Szabó has shown in [26] that such crossed products have finite nuclear dimension. Since the action is free, it is not hard to see that every ideal of the crossed product is generated by an ideal of $C(X)$ (cf. [25] for example); since $C(X)$ has the ideal property, it follows that A does as well.

We note that for C*-algebras of the form in (b), the condition of no elementary subquotients (which is of course necessary for $\mathcal{Z}$-stability) is equivalent to the following: there is no pair of α-invariant closed subsets $Y, Z \subseteq X$ such that $Y \setminus Z$ is nonempty and (at most) countable.

Let us prepare now to prove Theorem 7.9, which will be done by applying Theorem 7.8. The following lemma clarifies the role of the condition (ii) in Theorem 7.9.

Lemma 7.11. *Let A be a* C**-algebra such that the topology of* Prim(A) *has a basis of compact open sets. The for each $\varepsilon > 0$ the set of elements $c \in A$ such that $(c-\varepsilon)_+$ is pseudocompact is dense in A_+.*

Proof. Let $a \in A_+$ and $\varepsilon > 0$. Let $I = \mathrm{Ideal}((a-\varepsilon)_+)$. Let us write I as supremum of compact ideals. Since the sum of compact ideals is compact, we can assume that this supremum is upward directed. Let (I_λ) be increasing with supremum I. Then $\mathrm{her}((a-\varepsilon)_+) = \overline{\bigcup_\lambda I_\lambda \cap \mathrm{her}((a-\varepsilon)_+)}$. Let us find $b \in \mathrm{her}(a-\varepsilon)_+ \cap I_\lambda$ that is close to $(a-\varepsilon)_+$. Let us assume that b generates I_λ. Now define $c = a - (a-\varepsilon)_+ + b$. Then $(c-\varepsilon)_+ = b$ and c is close to a. □

Lemma 7.12. *Let A be a* C**-algebra of locally finite nuclear dimension. Suppose that the topology of* Prim(A) *has a basis of compact open sets. Let $F \subset A_+$ be a finite set of contractions, and let $\varepsilon, \gamma > 0$. Then there exists a* C**-subalgebra $B \subseteq A$ of finite nuclear dimension such that for each $c \in F$ there exists $c' \in B_+$ such that $c \approx_\gamma c'$ and $[(c'-\varepsilon)_+]$ is pseudocompact.*

Proof. By the previous lemma, we can find a finite set F' such that $F \subseteq_{\frac{\gamma}{2}} F'$ and for each $c \in F'$ we have that $[(c-\varepsilon)_+]$ is a pseudocompact element of $\mathrm{Cu}(A)$. Let $t_0 \in (0, \frac{\gamma}{2})$ be such that, for each $c' \in F'$ we have that $[(c'-\varepsilon)_+] \propto [(c'-\varepsilon-t_0)_+]$. If $c'' \in A_+$ is such that $c'' \approx_{\frac{t_0}{3}} c'$ then

$$[(c''-\varepsilon-\frac{t_0}{3})_+] \leqslant [(c'-\varepsilon)_+] \propto [(c'-\varepsilon-t_0)_+] \leqslant [(c''-\varepsilon-\frac{2t_0}{3})_+].$$

Thus, $(c''-\varepsilon-\frac{t_0}{3})_+$ is pseudocompact. Let us find $B \subseteq A$, of finite nuclear dimension, such that for each $c' \in F'$ there exists $c'' \in B$ such that $c'' \approx_{\frac{t_0}{3}} c'$. Set $c''' = (c''-\frac{t_0}{3})_+ \in B$. Then $[(c'''-\varepsilon)]_+$ is pseudocompact and $c''' \approx_\gamma c$. □

Lemma 7.13. *Let A be as in Theorem 7.9. Then there exist orthogonal positive elements $d^0, d^1 \in \mathrm{F}(A)$ such that $1 \preceq_A d^i \otimes 1_{3(M+1)}$ for $i = 0, 1$ (where $\preceq_A$ is as defined after Lemma 5.2).*

Proof. By the remark following Lemma 5.2, it suffices to show that there exist orthogonal elements $d^0, d^1 \in \mathrm{F}(A)$ and $\delta > 0$ such that, for each contraction $c \in A_+$,

$$[(c-\frac{1}{2})_+] \leqslant 3(M+1)[(cd^i-\delta)_+].$$

Our δ will be $\frac{1}{6}$.

It suffices by a diagonal sequence argument to show that for each finite set $F \subset A_+$ of positive contractions and $\gamma > 0$ there exist $d^0, d^1 \in A_\omega$ such that $\|[d^i, F]\| < \gamma$ and

$$[(c-\frac{1}{2})_+] \leqslant 3(M+1)[((d^i)^{1/2}c(d^i)^{1/2}-\delta)_+]$$

for all $c \in F$. By the previous lemma, there exists $B \subseteq A$ of finite nuclear dimension such that for each $c \in F$ there exists $c' \in B$ such that $c \approx_{\frac{1}{6}} c'$ and $[(c'-\frac{1}{3})_+]$ is pseudocompact.

By Theorem 6.1, there exist two orthogonal elements $d^0, d^1 \in \mathrm{F}(B, A)$ such that $d_\tau(d^i) > \frac{1}{3}d_\tau(1)$ for all bounded quasitraces τ on $\mathrm{F}(B, A)$. By Lemma 6.2, there exist $L, p, q \in \mathbb{N}$ such that $\frac{p}{q} > \frac{1}{3}$ and $L[1] + p[1] \leqslant L[1] + q[d^i]$ in $\mathrm{Cu}(\mathrm{F}(B, A))$ for $i = 0, 1$. (Although it is unimportant to the argument here, the proof of Lemma 6.7 shows why we may use the same values for both d^0 and d^1.) Let $\varepsilon > 0$ be such that $L[1]+p[1] \leqslant L[1]+q[(d^i-\varepsilon)_+]$ for $i = 0, 1$; without loss of generality (by possibly modifying d^i by functional calculus), we may assume that $\varepsilon = \frac{1}{3}$.

It follows that $L[b] + p[b] \leqslant L[b] + q[(d^i-\frac{1}{3})_+b]$ in $\mathrm{Cu}(A_\omega)$, for all $b \in B_+$. When $[b]$ is pseudocompact, we get that $[b] \leqslant 3(M+1)[(d^i-\frac{1}{3})_+b]$, by Proposition 2.2 and Lemma 7.3 (ii).

Thus, for $c \in F$,

$$\begin{aligned}
[(c-\frac{1}{2})_+] &\leqslant [(c'-\frac{1}{3})_+] \\
&\leqslant 3(M+1)[(d^i-\frac{1}{3})_+(c'-\frac{1}{3})_+] \\
&\leqslant 3(M+1)[((d^i)^{1/2}c'(d^i)^{1/2}-\frac{1}{3})_+] \\
&\leqslant 3(M+1)[((d^i)^{1/2}c(d^i)^{1/2}-\frac{1}{6})_+],
\end{aligned}$$

where on the third line, we used Lemma 5.1, and on the last line we used the fact that $(d^i)^{1/2}c'(d^i)^{1/2} \approx_{\frac{1}{6}} (d^i)^{1/2}c(d^i)^{1/2}$. □

Proof of Theorem 7.9. Let $B \subseteq A_\omega$ be a separable C*-subalgebra of nuclear dimension at most m. By Lemma 7.13 and a speeding-up argument (such as in the proof of [37, Proposition 4.4]), there exist orthogonal full elements $d^0, d^1 \in \mathrm{F}(B,A)$ such that $1 \precsim_B d^i \otimes 1_{3(M+1)}$ for $i = 0, 1$. Thus, by Proposition 5.3,

$$[1] \leqslant 3(M+1)(2m+2)[d^i]$$

in $\mathrm{Cu}(\mathrm{F}(B,A))$, for $i = 0, 1$.

Thus, the hypothesis of Theorem 7.8 is satisfied with $P_m := 6(M+1)(m+1)$; consequently, A is $\mathcal{Z}$-stable. □

7.4. **C*-algebras with Hausdorff spectrum and finite quotients.** Here we show the following:

Theorem 7.14. *Let A be a separable* C*-*algebra of locally finite nulear dimension which is (M,N)-pure for some $M, N \in \mathbb{N}$. Suppose also that*

(i) no nonzero simple quotient of A is purely infinite; and
(ii) the primitive ideal space of A is Hausdorff.

Then A is $\mathcal{Z}$-stable.

Combined with Theorem 3.1, the preceding theorem yields at once the following

Corollary 7.15. *Let A be a separable* C*-*algebra of finite nuclear dimension with no type I quotients. Suppose also that A satisfies conditions (i) and (ii) of the previous theorem. Then A is $\mathcal{Z}$-stable.*

Theorem 7.14 may be proven by a slight adjustment to the proof in the simple case. We must generalize Proposition 7.5 as follows.

Proposition 7.16. *Let A be a* C*-*algebra with Hausdorff primitive ideal space, such that no nonzero simple quotient of A is purely infinite, and suppose that A has M-comparison for some M. Let $B \subseteq A$ be a* C*-*subalgebra of nuclear dimension at most m. Then in every quotient of* $\mathrm{F}(B,A)$, 1 *is not stably properly infinite.*

Proof. This proof is an adaptation of the proof of Proposition 7.5. For a contradiction, suppose that in some nonzero quotient of $\mathrm{F}(B,A)$, we have $(k+1)[1] \leqslant k[1]$. Equivalently, there exists a non-full element $[b] \in \mathrm{Cu}(\mathrm{F}(B,A))$ such that $(k+1)[1] \leqslant k[1] + [b]$. Consequently, for some $\varepsilon > 0$, $(k+1)[1] \leqslant k[1] + [(b-\varepsilon)_+]$.

For $c \in B_+$ and $\eta > 0$, we wish to show that $[(c-\eta)_+] \leqslant (M+1)[(b-\varepsilon)_+(c-\frac{\eta}{2})_+]$ in $\mathrm{Cu}(A_\omega)$. We know that

$$(k+1)[(c-\frac{\eta}{2})_+] \leqslant k[(c-\frac{\eta}{2})_+] + [(b-\varepsilon)_+(c-\frac{\eta}{2})_+] \tag{7.1}$$

in $\mathrm{Cu}(A_\omega)$.

Set $X := \mathrm{Prim}(A)$ and let us regard A as a $C_0(X)$-algebra in the natural way. Set $K := \{x \in X \mid \|c(x)\| \geqslant \eta\}$. Then $(c-\frac{\eta}{2})_+$ is a pseudocompact element of $\mathrm{Cu}(A_K)$, and consequently it is also pseudocompact in $\mathrm{Cu}((A_K)_\omega)$. By Lemma 7.4, and Lemma 7.3 (ii) applied to (7.1), we have

$$[(c-\frac{\eta}{2})_+] \leqslant (M+1)[(b-\varepsilon)_+(c-\frac{\eta}{2})_+]$$

in $\mathrm{Cu}((A_K)_\omega)$, and therefore, $[(c-\eta)_+] \leqslant (M+1)[(b-\varepsilon)_+(c-\frac{\eta}{2})_+]$ in $\mathrm{Cu}((A_K)_\omega)$. Since the quotient map $A_\omega \to (A_K)_\omega$ is an isomorphism on $\mathrm{Ideal}((c-\eta)_+)$, it follows that $[(c-\eta)_+] \leqslant (M+1)[(b-\varepsilon)_+(c-\frac{\eta}{2})_+]$ in $\mathrm{Cu}(A_\omega)$, as desired.

It now follows by Proposition 5.3 that

$$[1] \leqslant 2(m+1)(M+1)[b],$$

which is a contradiction, since $[b]$ is not full. □

Proof of Theorem 7.14. Proceed exactly as in the proof of Theorem 7.1, using Proposition 7.16 in place of Proposition 7.5. □

Let us discuss the relevance of these results. For C*-algebras A as in Theorem 7.14, the main result of [27] says that the simple quotients of A are all $\mathcal{Z}$-stable. Moreover, Hirshberg, Rørdam, and Winter showed in [7, Theorem 4.6] that if the primitive ideal space of A has finite covering dimension and all its simple quotients are $\mathcal{Z}$-stable, then A is $\mathcal{Z}$-stable; thus, Theorem 7.14 only says something new in the case that the primitive ideal space of A is infinite dimensional. Examples of Hirshberg-Rørdam-Winter and of Dadarlat show a variety of possibilities for C*-algebras with infinite-dimensional, Hausdorff primitive ideal space and $\mathcal{Z}$-stable simple quotients [7, Examples 4.7 and 4.8], [3, Section 3]. Our result, combined with results of the second-named author and Winter in [28], neatly characterizes $\mathcal{Z}$-stability for such C*-algebras:

Corollary 7.17. *Let A be a finite C*-algebra with Hausdorff primitive ideal space and no type I quotients. Then the following are equivalent.*

(i) A is $\mathcal{Z}$-stable, and there is a finite bound on the decomposition rank of the simple quotients of A;
(ii) A has finite decomposition rank.

Proof. (i) ⇒ (ii) follows from [28, Theorem 4.1 and Lemma 6.1]. (ii) ⇒ (i) follows from Corollary 7.15. □

The equivalence of the following two conditions, under the hypothesis of the above corollary, would follow from conjecture (C1):

(i') A is $\mathcal{Z}$-stable, and there is a finite bound on the nuclear dimension of the simple quotients of A;
(ii') A has finite nuclear dimension.

(i') ⇒ (ii') follows from [28, Theorem 4.1 and Lemma 6.1]. In the case that every simple quotient is infinite, an implication similar to (but stronger than) (ii') ⇒ (i') has been considered by Blanchard, Kirchberg, and Rørdam in [2, 11]. In particular, using results of [11, Theorem 8.6] and [2, Theorem 5.8], it suffices to show in this case that the C*-algebra has 0-comparison.

More generally, note that if a C*-algebra A has Hausdorff primitive ideal space X, then the set of points $x \in X$ corresponding to infinite simple quotients forms an open set. This is because: a C*-algebra is infinite if and only if it contains a partial isometry v such that $v^*v < vv^*$, and this is a stable relation. Therefore, A is an extension of the two cases (all simple quotients are infinite, and no simple quotients are infinite), and thus, the general case reduces to the case that every quotient is infinite.

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Department of Mathematics University of Louisiana at Lafayette Lafayette, USA.
E-mail address: lrobert@louisiana.edu

Aaron Tikuisis, Institute of Mathematics, University of Aberdeen, Aberdeen, United Kingdom.
E-mail address: a.tikuisis@abdn.ac.uk