Central sequences, dimension, and $\mathcal{Z}$-stability of C^*-algebras

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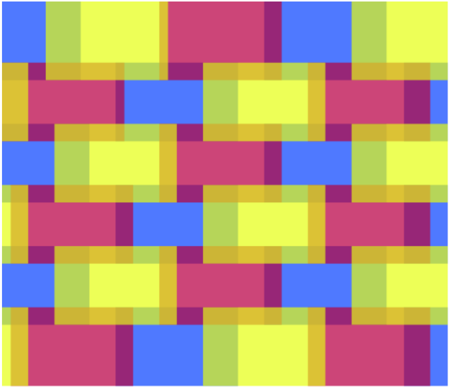
Groups, dynamical systems, and C^*-algebras

Dimension

Nuclear dimension generalizes covering dimension to C^*-algebras



Comes naturally by treating approximations in the completely positive approximation property as **non-commutative partitions of unity.**

Dimension

$\dim X \leq n$ iff

$$\begin{array}{ccc} C(X) & \xrightarrow{\quad = \quad} & C(X) \\ & {\scriptstyle f\mapsto(f(x_j^{(i)}))} \searrow \qquad \nearrow {\scriptstyle (\lambda_j^{(i)})\mapsto\sum_{i,j}\lambda_j^{(i)}e_j^{(i)}} & \\ & \bigoplus_{i,j}\mathbb{C} & \end{array}$$

commuting pointwise-$\|\cdot\|$ approximately, where $x_j^{(i)} \in X$ and $(e_j^{(i)})_{i=0,\dots,n;\, j=1,\dots,r}$ is an $(n+1)$-colourable partition of unity, ie. $e_1^{(i)}, \dots, e_r^{(i)}$ are orthogonal for each colour i.

Dimension

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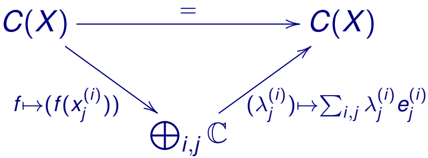
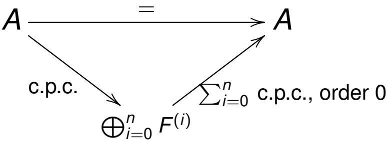


commuting pointwise-$\| \cdot \|$ approximately.

Definition (Winter-Zacharias '09, Kirchberg-Winter '02)
Nuclear dimension $\leq n$:



Commuting pointwise-$\| \cdot \|$ approximately; $F^{(i)}$ is f.d.

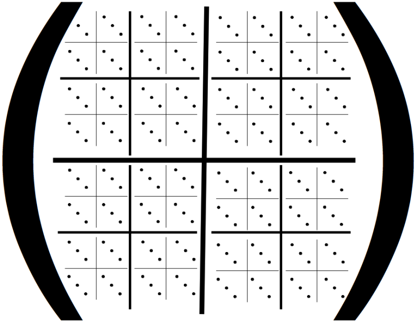
Order 0 means orthogonality preserving,
$ab = 0 \Rightarrow \phi(a)\phi(b) = 0$.

The Jiang-Su algebra $\mathcal{Z}$ is a C^*-algebra which:

- is self-absorbing ($\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$);
- has a lot of uniformity: any unital *-homomorphism $\mathcal{Z} \to \mathcal{Z}$ is approximately inner;
- makes good things happen to C^*-algebras by $\otimes$.

The Jiang-Su algebra

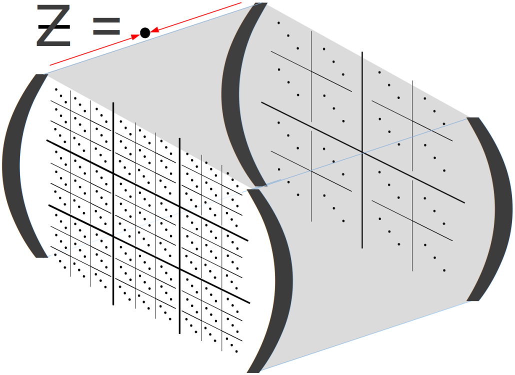
UHF algebras:



M_{2^∞}

The Jiang-Su algebra

Jiang-Su algebra:



Nuclear dimension and $\mathcal{Z}$-stability

Shocking conjecture: finite nuclear dimension coincides with $\mathcal{Z}$-stability for nuclear, separable C^*-algebras with no type I subquotients.

Assuming simplicity:
($\Rightarrow$) is was shown by Winter '10 (and T '12, nonunital case).

($\Leftarrow$) was shown by Matui-Sato '13, assuming at most one trace (and quasidiagonal if finite).

($\Leftarrow$) also occurs by classification, eg. assuming rationally tracial rank one (Lin '08).

Without simplicity, ($\Leftarrow$) holds for AH algebras (T-Winter '12).

The central sequence algebra

Let A be unital (from now on).

$$A_\infty := \ell_\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A).$$

A sits inside A_∞ as constant sequences.

The central sequence algebra is

$$A_\infty \cap A'$$

A McDuff-type theorem for $\mathcal{Z}$-stability

$$A_\infty := \ell_\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A).$$

Theorem (Kirchberg, Rørdam '90's)

A separable C^*-algebra A is $\mathcal{Z}$-stable if and only if there is a unital *-homomorphism $\mathcal{Z} \to A_\infty \cap A'$.

(McDuff's Theorem '69: M is $\mathcal{R}$-stable iff there is a unital *-homomorphism $M \to M_\infty \cap M'$.)

(In nonunital case, use $(A_\infty \cap A')/\{x \in A_\infty \mid xA = Ax = 0\}$ instead of $A_\infty \cap A'$.)

The Cuntz semigroup

For $a, b \in (B \otimes \mathcal{K})_+$, $[a] \leq [b]$ means that there exists $(d_n) \subset B \otimes \mathcal{K}$ such that

$$d_n^* b d_n \to a.$$

Set $\mathcal{C}u(B) := \{[a] \mid a \in (B \otimes \mathcal{K})_+\}$.

For $[a], [b] \in \mathcal{C}u(B)$, we set

$$[a] + [b] := [a \oplus b]$$

(using $M_2 \otimes \mathcal{K} \cong \mathcal{K}$).

The Cuntz semigroup of $A_\infty \cap A'$

$[a] \leq [b]$ if $\exists (d_n)$ such that $d_n^* b d_n \to a$.

Proposition (Rørdam-Winter '08)

There is a unital *-homomorphism $\mathcal{Z} \to A_\infty \cap A'$ (ie. A is $\mathcal{Z}$-stable) if and only if, for any k, there exists a c.p.c. order zero map $\phi : M_k \to A_\infty \cap A'$ such that

$$[1 - \phi(1_n)] \leq [(\phi(e_{11}) - \delta)_+]$$

in $\mathcal{C}u(A_\infty \cap A')$, for some $\delta > 0$.

Proposition

There is a unital *-homomorphism $\mathcal{Z} \to A_\infty \cap A'$ if and only if $\mathcal{C}u(A_\infty \cap A')$ has M-comparison and $\mathcal{C}u(A_\infty \cap A')$ is N-almost divisible for some $M, N \in \mathbb{N}$.

Definitions of M-comparison and N-almost divisible

Proposition

There is a unital *-homomorphism $\mathcal{Z} \to A_\infty \cap A'$ if and only if $\mathcal{C}u(A_\infty \cap A')$ has M-comparison and $\mathcal{C}u(A_\infty \cap A')$ is N-almost divisible for some $M, N \in \mathbb{N}$.

$\mathcal{C}u(B)$ has M-comparison if whenever $[a], [b_0], \dots, [b_M] \in \mathcal{C}u(A)$ satisfy $(k+1)[a] \leq k[b_i]$ for some $k \in \mathbb{N}$, it follows that

$$[a] \leq [b_0] + \cdots + [b_M].$$

$\mathcal{C}u(B)$ is N-almost divisible if whenever $[a] \ll [a'] \in \mathcal{C}u(B)$ and $k \in \mathbb{N}$, there exists $[x] \in \mathcal{C}u(B)$ such that

$$k[x] \leq [a'] \quad \text{and} \quad [a] \leq (N+1)(k+1)[x].$$

Nuclear dimension and the Cuntz semigroup

Proposition

There is a unital *-homomorphism $\mathcal{Z} \to A_\infty \cap A'$ if and only if $\mathcal{C}u(A_\infty \cap A')$ has M-comparison and $\mathcal{C}u(A_\infty \cap A')$ is N-almost divisible for some $M, N \in \mathbb{N}$.

Theorem (Robert '10)

If $\dim_{nuc} A \leq n$ then A has n-comparison.

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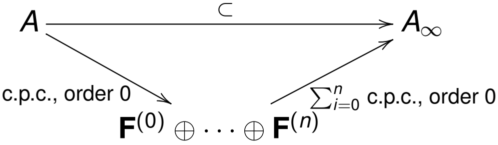
Proof:



commuting exactly, where $\mathbf{F}^{(i)} = \prod_j F_j^{(i)} / \bigoplus_j F_j^{(i)}$ and $F_j^{(i)}$ is f.d.

$\mathbf{F}^{(i)}$ has 0-comparison, so this shows A has n-comparison.

Nuclear dimension and the Cuntz semigroup

Proposition

If $\dim_{nuc} A \leq n$, $a \in A_+$, and $\overline{aAa}$ has no type I subquotients and has $(n+1)$ orthogonal full elements then $[a] \in \mathcal{C}u(A)$ is m-almost divisible, for some m.

Almost divisibility **isn't free.**

It entails the "global Glimm property" (and particular, orthogonal full elements).

Nuclear dimension and central sequences

Theorem (Robert-T, '13)

Let A have finite nuclear dimension. Then

$$\begin{array}{ccc} A_\infty \cap A' & \xrightarrow{\quad\subset\quad} & (A_\infty)_\infty \cap A' \\ & {\scriptstyle \text{c.p.c., order } 0} \searrow \qquad \nearrow {\scriptstyle \sum_{i=0}^{N} \text{c.p.c., order } 0} & \\ & \mathbf{C}^{(0)} \oplus \cdots \oplus \mathbf{C}^{(N)} & \end{array}$$

commuting exactly, where $\mathbf{C}^{(i)}$ is a hereditary subalgebra of $(A_\infty)_\infty$.

Here, $N = 2\dim_{nuc} A + 1$.

Since $\mathcal{C}u(A)$ has n-comparison, so does $\mathbf{C}^{(i)}$.

This shows that $\mathcal{C}u(A_\infty \cap A')$ has $((N+1)(n+1)-1)$-comparison.

Nuclear dimension and central sequences

Theorem (Robert-T, '13)

Let A have finite nuclear dimension.

$$\begin{array}{ccc} A_\infty \cap A' & \xrightarrow{\quad\subset\quad} & (A_\infty)_\infty \cap A' \\ & \searrow^{\text{c.p.c., order 0}} \quad\quad \nearrow_{\sum_{i=0}^{N} \text{c.p.c., order 0}} & \\ & \mathbf{C}^{(0)} \oplus \cdots \oplus \mathbf{C}^{(N)} & \end{array}$$

commuting exactly.

If $\mathcal{C}u(A)$ is N-almost divisible, then we can prove a weaker divisibility property for $\mathcal{C}u(A_\infty \cap A')$ (but not $\mathcal{Z}$-stability).

Theorem (Robert-T, '13)

If A is separable and $\dim_{nuc} A < \infty$, then A is $\mathcal{Z}$-stable if and only if $A_\infty \cap A'$ has two orthogonal full elements.

New results

Theorem (Robert-T, '13)

If A has finite nuclear dimension, no type I subquotients, no purely infinite subquotients, and $\mathrm{Prim}(A)$ has a basis of compact open sets, then A is $\mathcal{Z}$-stable.

Eg. if A has finite decomposition rank and real rank zero.

Eg. if $A = C(X) \rtimes \mathbb{Z}^n$, where X is the Cantor set and the action is free; $\dim_{nuc} A < \infty$ thanks to Szabó.

New results

Theorem (Robert-T, '13)

If A has finite nuclear dimension, no type I subquotients, no purely infinite quotients, and $\mathrm{Prim}(A)$ is Hausdorff, then A is $\mathcal{Z}$-stable.

Note: $\mathrm{Prim}(A)$ may be infinite-dimensional (eg. $\dim_{nuc} C(X, \mathcal{Z}) \leq 2$, where X is the Hilbert cube).

Corollary (Robert-T '13, T-Winter '12)

If A is a $C_0(X)$-algebra, all of whose fibres are simple, then A has finite decomposition rank if and only if A is $\mathcal{Z}$-stable and the fibres have bounded decomposition rank.

Outlook

Question

If A has no type I subquotients, does it have two orthogonal almost full elements?
Does it help to assume $\dim_{nuc} A < \infty$?

Questions about nice C^*-algebras ($\mathcal{Z}$-stable or $\dim_{nuc} < \infty$):

Question

What does $\mathcal{C}u(A_\infty \cap A')$ look like? Even for $A = \mathcal{Z}$?

Question

If $a \in A_\infty \cap A'$ is full in A_∞, is it full in $A_\infty \cap A'$?