Central sequences, dimension, and $\mathcal{Z}$-stability of C^*-algebras

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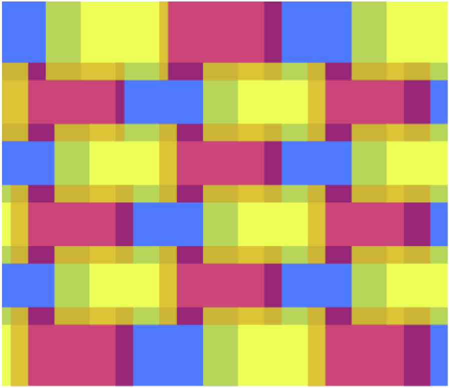
C^*-Algebren

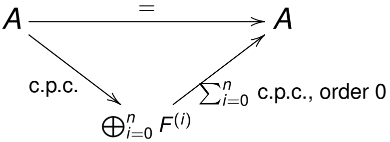
Dimension

Nuclear dimension generalizes covering dimension to C^*-algebras



Comes naturally by treating approximations in the completely positive approximation property as **non-commutative partitions of unity.**

Nuclear dimension $\leq n$:



Commuting pointwise-$\|\cdot\|$ approximately; $F^{(i)}$ is f.d.

Order 0 means orthogonality preserving,
$ab = 0 \Rightarrow \phi(a)\phi(b) = 0$.

The Jiang-Su algebra

The Jiang-Su algebra $\mathcal{Z}$ is a C^*-algebra which:

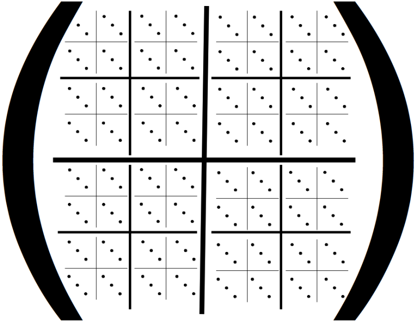
• is self-absorbing ($\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$);

• has a lot of uniformity: any unital *-homomorphism $\mathcal{Z} \to \mathcal{Z}$ is approximately inner;

• makes good things happen to C^*-algebras by $\otimes$ (eg. classification);

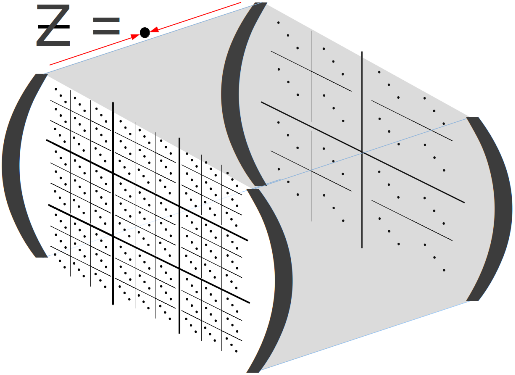
• has the K-theory and traces of $\mathbb{C}$, so $\mathcal{Z}$-stability is not *too* restrictive.

UHF algebras:



M_{2^∞}

Jiang-Su algebra:



Nuclear dimension and $\mathcal{Z}$-stability

Shocking conjecture: finite nuclear dimension coincides with $\mathcal{Z}$-stability for nuclear, separable C^*-algebras with no type I subquotients.

Assuming simplicity:
($\Rightarrow$) is was shown by Winter '10 (and T '12, nonunital case).

($\Leftarrow$) was shown by Matui-Sato '13, assuming at most one trace (and quasidiagonal if finite).

($\Leftarrow$) also occurs by classification, eg. assuming rationally tracial rank one (Lin '08).

Without simplicity, ($\Leftarrow$) holds for AH algebras (T-Winter '12).

The central sequence algebra

Let A be unital (from now on).

The sequence algebra: $A_\infty := \prod_{n=1}^\infty A / \oplus_{n=1}^\infty A$.

A sits inside A_∞ as constant sequences.

The central sequence algebra: $A_\infty \cap A'$.

A McDuff-type theorem for $\mathcal{Z}$-stability

$$A_\infty := \ell_\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A).$$

Theorem (Kirchberg, Rørdam '90's)

A separable C^*-algebra A is $\mathcal{Z}$-stable if and only if there is a unital *-homomorphism $\mathcal{Z} \to A_\infty \cap A'$.

(McDuff's Theorem '69: M is $\mathcal{R}$-stable iff there is a unital *-homomorphism $\mathcal{R} \to M_\infty \cap M'$.)

(In nonunital case, use $(A_\infty \cap A')/\{x \in A_\infty \mid xA = Ax = 0\}$ instead of $A_\infty \cap A'$.)

The Cuntz semigroup of $A_\infty \cap A'$

Recall: $[a] \leq [b]$ if $\exists (d_n)$ such that $d_n^* b d_n \to a$;
$\mathcal{C}u(A) = \{[a] : a \in (A \otimes \mathcal{K})_+\}$.

Proposition (Rørdam-Winter '08)

There is a unital *-homomorphism $\mathcal{Z} \to A_\infty \cap A'$ (ie. A is $\mathcal{Z}$-stable) if and only if the $\mathcal{C}u(A_\infty \cap A')$ is nice in the following sense:

(i) $\mathcal{C}u(A_\infty \cap A')$ is almost unperforated (order determined by traces); and

(ii) $\mathcal{C}u(A_\infty \cap A')$ is almost divisible.

In fact (i) can be weakened to M-comparison and (ii) to $\mathcal{C}u(A_\infty \cap A')$ being N-almost divisible.

Nuclear dimension and the Cuntz semigroup

Proposition

There is a unital *-homomorphism $\mathcal{Z} \to A_\infty \cap A'$ if and only if $\mathcal{C}u(A_\infty \cap A')$ has M-comparison and $\mathcal{C}u(A_\infty \cap A')$ is N-almost divisible for some $M, N \in \mathbb{N}$.

Theorem (Robert '10)

If $\dim_{nuc} A \leq n$ then $\mathcal{C}u(A)$ has n-comparison.

Nuclear dimension and the Cuntz semigroup

Proposition

If $\dim_{nuc} A \leq n$ then $\mathcal{C}u(A)$ is n-almost divisible.

At a minimum, we need to assume no type I subquotients.

It entails the "global Glimm property" (and particular, orthogonal full elements).

Nuclear dimension and central sequences

Theorem (Robert-T, '13)

Let A have finite nuclear dimension. Then

$$A_\infty \cap A' \xrightarrow{\ \subset\ } (A_\infty)_\infty \cap A'$$

$$A_\infty \cap A' \xrightarrow{\text{c.p.c., order } 0} \mathbf{C}^{(0)} \oplus \cdots \oplus \mathbf{C}^{(N)} \xrightarrow{\sum_{i=0}^{N} \text{ c.p.c., order } 0} (A_\infty)_\infty \cap A'$$

commuting exactly, where $\mathbf{C}^{(i)}$ is a hereditary subalgebra of $(A_\infty)_\infty$.

Here, $N = 2\dim_{nuc} A + 1$.

Since $\mathcal{C}u(A)$ has n-comparison, so does $\mathbf{C}^{(i)}$.

Corollary

$\mathcal{C}u(A_\infty \cap A')$ has $((N+1)(n+1)-1)$-comparison.

Nuclear dimension and central sequences

Theorem (Robert-T, '13)

Let A have finite nuclear dimension. Then

$$\begin{array}{ccc} A_\infty \cap A' & \xrightarrow{\quad\subset\quad} & (A_\infty)_\infty \cap A' \\ & \searrow^{\text{c.p.c., order } 0} \qquad \nearrow_{\sum_{i=0}^{N} \text{ c.p.c., order } 0} & \\ & \mathbf{C}^{(0)} \oplus \cdots \oplus \mathbf{C}^{(N)} & \end{array}$$

commuting exactly.

Theorem (Robert-T, '13)

If A is separable and $\dim_{nuc} A < \infty$, then A is $\mathcal{Z}$-stable if and only if $A_\infty \cap A'$ has two orthogonal full elements.

New results

Theorem (Robert-T, '13)

If A has finite nuclear dimension, no type I subquotients, no purely infinite subquotients, and $\mathrm{Prim}(A)$ has a basis of compact open sets, then A is $\mathcal{Z}$-stable.

Eg. if A has finite decomposition rank and real rank zero.

Eg. if $A = C(X) \rtimes \mathbb{Z}^n$, where X is the Cantor set and the action is free; $\mathrm{dim}_{nuc} A < \infty$ thanks to Szabó.

New results

Theorem (Robert-T, '13)

If A has finite nuclear dimension, no type I subquotients, no purely infinite quotients, and $\mathrm{Prim}(A)$ is Hausdorff, then A is $\mathcal{Z}$-stable.

Note: $\mathrm{Prim}(A)$ may be infinite-dimensional (eg. $\dim_{nuc} C(X, \mathcal{Z}) \leq 2$, where X is the Hilbert cube).

Corollary (Robert-T '13, T-Winter '12)

If A is a $C_0(X)$-algebra, all of whose fibres are simple, then A has finite decomposition rank if and only if A is $\mathcal{Z}$-stable and the fibres have bounded decomposition rank.

Outlook

Question

If A has no type I subquotients, does it have two orthogonal almost full elements?
Does it help to assume $\dim_{nuc} A < \infty$?

Questions about nice C^*-algebras ($\mathcal{Z}$-stable or $\dim_{nuc} < \infty$):

Question

If $a \in A_\infty \cap A'$ is full in A_∞, is it full in $A_\infty \cap A'$? Even for A strongly purely infinite?

Question

What does $\mathcal{C}u(A_\infty \cap A')$ look like? Even for $A = \mathcal{Z}$?