Dimension reduction and Jiang-Su stability

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(joint work with Wilhelm Winter)

Recent developments in the study of classification of C^*-algebras have suggested an important role of new regularity conditions (more stringent than amenability). These developments arise in response to an example of Villadsen [7], which was built on by Rørdam [3] (cf. also [5]), to disprove the Elliott conjecture. The general idea is captured by the following conjecture:

Conjecture. *For a simple, separable, unital, nonelementary, nuclear C^*-algebra A in the UCT class, the following are equivalent:*

(i) *A is $\mathcal{Z}$-stable;*
(ii) *A has finite nuclear dimension;*
(iii) *A has strict comparison of positive elements;*
(iv) *A is an inductive limit of nice building blocks (2-NCCW complexes, direct sums of $M_n \otimes \mathcal{O}_m \otimes C(\mathbb{T})$).*

Moreover, the algebras satisfying (i)-(iv) are classifiable.

(Closely related is the Toms-Winter conjecture, stating that (i),(ii), and (iii) are equivalent even without assuming the UCT.) It should be noted that the conjecture is known to hold for the examples of Villadsen [6].

This talk focused on the relationship between properties (i) and (ii), although it is important to view their relationship in context of the other two properties. A C^*-algebra is said to be $\mathcal{Z}$-stable if it is isomorphic to its tensor product with $\mathcal{Z}$. Nuclear dimension is a non-commutative generalization of topological dimension, building on the idea that the completely positive approximation property is a noncommutative version of (arbitrarily fine) partitions of unity [9].

Among the many partial verifications of the conjecture, we note that (ii) $\Rightarrow$ (i) has been shown by Winter in full generality [8]. On the other hand, (i) $\Rightarrow$ (ii) is perhaps the least-understood implication of the conjecture, and earlier verifications of this implication have always relied on classification (i.e. factored through (iv)).

The following result is, we hope, the beginning of a new approach to establishing and understanding (i) $\Rightarrow$ (ii):

Theorem. *(T-Winter [4]) The decomposition rank of $C(X, \mathcal{Z})$ is at most 2, independent of X.*

This result, and (i) $\Rightarrow$ (ii) in general, amounts to dimension reduction: showing that tensoring with the Jiang-Su algebra has the effect of lowering the dimension (at least, when the dimension is sufficiently high beforehand). Some notable earlier results about dimension reduction are the following: Villadsen's example A (mentioned above) has infinite nuclear dimension, but by classification, $A \otimes \mathcal{Z}$ has decomposition rank at most 2. Gong's reduction theorem [1] states that, if A is a simple AH algebra with very slow dimension growth then it is a limit of algebras with topological dimension at most three. Finally, Kirchberg and Rørdam [2] showed that for any space X, $C_0(X, \mathbb{C}\cdot 1_{\mathcal{O}_2}) \subset C(X, \mathcal{O}_2)$ factors as

$$C_0(X) \to C_0(Y) \to C(X, \mathcal{O}_2),$$

where $\dim Y \leq 1$. The latter result highly relies on $K_*(\mathcal{O}_2) = 0$ (and little else), and is used in the proof of the result of myself and Winter mentioned above.

Part of this talk concerned explaining some key ideas from the proof of the result of myself and Winter. A key point of this proof is establishing the following:

Lemma. *Let $X = [0,1]^d$. Then $C(X, \mathbb{C}\cdot 1_{n^\infty}) \subset C(X, M_{n^\infty})$ can be approximately factorized as*

$$C(X) \xrightarrow{\psi} C_0(Y, \mathbb{C}\cdot 1_{\mathcal{O}_2}) \oplus F \subset C_0(Y, \mathcal{O}_2) \oplus F \xrightarrow{\phi} C(X, M_{n^\infty}),$$

where ψ, ϕ are c.p.c. and ϕ is order zero when restricted to $C_0(Y, \mathcal{O}_2)$ or F.

In fact, the result follows (at least with M_{n^∞} in place of $\mathcal{Z}$) from this and Kirchberg-Rørdam's result for $C_0(Y) \subset C_0(Y, \mathcal{O}_2)$. The lemma is proven somewhat explicitly for the case $d = 1$ (using as input a c.p.c. approximate embedding of $C_0((0,1], \mathcal{O}_2)$ to M_{n^k}, given by quasidiagonality of the cone over $\mathcal{O}_2$), and then for general d roughly by taking products.

The result on dimension reduction opens many questions, including the following: Can we say more about the structure of $C(X) \subset C(X, \mathcal{Z})$; does it (approx.) factorize through subhomogeneous algebras with low topological dimension? Is $\dim_{\mathrm{nuc}}(A \otimes \mathcal{Z}) < \infty$ for every nuclear C^*-algebra A? What is the decomposition rank of $C(X) \subset (X, M_n)$; is it $< \dim X$, or does this dimension drop only occur when we put a UHF algebra in for M_n?

Slides from the talk may be found on my website, `http://www.math.uni-muenster.de/u/aaron.tikuisis`.

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