MAXIMALLY UNITARILY MIXED STATES ON A C*-ALGEBRA

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1. Preliminaries on Dixmier sets

Let A be a C^*-algebra. We denote by A_{sa} the set of self-adjoint elements of A and by A_+ the set of positive elements of A. Let A^* denote the dual of A. We denote by A^*_{sa} the set of self-adjoint functionals in A^* and by A^*_+ the set of positive functionals in A^*.

1.1. **Dixmier sets on A and A^*.** We call a set $C \subseteq A$ a Dixmier set if it is convex, norm-closed, and invariant under unitary conjugation. The latter means that $uCu^* \subseteq C$ for all unitaries $u \in A^\sim$ (where $A^\sim$ is the minimal unitization of A, i.e. A itself if A is unital, and the unitization $A + \mathbb{C}1$ if A is non-unital). We will largely work with singly generated Dixmier sets. Given $a \in A$ we denote by $D_A(a)$ the smallest Dixmier set containing a.

We let A, and more generally $M(A)$ (the multiplier algebra of A), act on A^* in the usual way: if $a \in M(A)$ and $\phi \in A^*$ then

$$a\phi(x) := \phi(ax), \quad (\phi a)(x) := \phi(xa) \qquad (x \in A).$$

A set $C \subseteq A^*$ is called a Dixmier set if it is convex, weak*-closed, and invariant under unitary conjugation. The latter condition means that $uCu^* \subseteq C$ for all unitaries u in $A^\sim$. Given $\phi \in A^*$ we denote by $D_A(\phi)$ the Dixmier set generated by ϕ, i.e., the smallest Dixmier set containing ϕ. Since $D_A(\phi)$ is weak*-closed and bounded, it is weak*-compact.

We shall make frequent use of the fact that A is the dual of A^* when the latter is endowed with the weak* topology [REF]. This, combined with the Hahn-Banach theorem, implies that elements of A separate disjoint weak*-compact convex sets in A^*.

Let $\mathcal{V}$ be a subgroup of the unitary group $\mathcal{U}(M(A))$ of $M(A)$. On some occasions we will need more general versions of the sets defined above where the unitaries range through $\mathcal{V}$ rather than all of $\mathcal{U}(A^\sim)$. Thus, given $a \in A$ we define $D_A(a, \mathcal{V})$ as the smallest norm-closed convex subset of A containing a and invariant under conjugation by unitaries in $\mathcal{V}$. Similarly, given $\phi \in A^*$ we define $D_A(\phi, \mathcal{V})$ as the the smallest weak*-closed convex subset of A^* containing ϕ and invariant under conjugation by unitaries in $\mathcal{V}$.

1.2. **Mixing operators.** Let $\mathcal{V}$ be a subgroup of the unitary group $\mathcal{U}(M(A))$ of $M(A)$. We call a linear operator $T\colon A \to A$ a $\mathcal{V}$-mixing operator if it is defined by an equation of the form

$$Ta = \sum_{j=1}^{n} \lambda_j u_j a u_j^* \qquad (a \in A),$$

where $n \in \mathbb{N}$, $\lambda_j > 0$, $u_j \in \mathcal{V}$ $(1 \leqslant j \leqslant n)$, and $\sum_{j=1}^n \lambda_j = 1$. Elementary properties of such operators are described in [2, 2.2]. We denote by $\mathrm{Mix}(A, \mathcal{V})$ the set of $\mathcal{V}$-mixing operators on A. If $\mathcal{V} = \mathcal{U}(A^\sim)$ we simply write $\mathrm{Mix}(A)$. Notice that

$$D_A(a, \mathcal{V}) = \overline{\{Ta : T \in \mathrm{Mix}(A, \mathcal{V})\}}^{\|\cdot\|}.$$

We also call an operator $T\colon A^* \to A^*$ a $\mathcal{V}$-mixing operator if it is the adjoint of a $\mathcal{V}$-mixing operator on A. In this case T has the form

$$T\phi = \sum_{j=1}^n \lambda_j u_j \phi u_j^* \qquad (\phi \in A^*),$$

where $n \in \mathbb{N}$, $\lambda_j > 0$, $u_j \in \mathcal{V}$ $(1 \leqslant j \leqslant n)$, and $\sum_{j=1}^n \lambda_j = 1$. Observe that T is positive ($T\phi \geqslant 0$ for all $\phi \geqslant 0$) and contractive. We denote the set of $\mathcal{V}$-mixing operators on A^* by $\mathrm{Mix}(A^*, \mathcal{V})$ or simply by $\mathrm{Mix}(A^*)$ if $\mathcal{V} = \mathcal{U}(A^\sim)$. Notice that

$$D_A(\phi, \mathcal{V}) = \text{weak*-cl}\{T\phi : T \in \mathrm{Mix}(A^*, \mathcal{V})\}.$$

Lemma 1.1. *Let $a \in A$ and $\phi \in A^*$. Then*

$$D_A(\phi, \mathcal{V})(a) = \overline{\phi(D_A(a, \mathcal{V}))}. \tag{1.1}$$

Proof. Since $D_A(\phi, \mathcal{V})$ is weak*-compact, $D_A(\phi, \mathcal{V})(a)$ is a closed subset of $\mathbb{C}$. To prove the lemma it suffices to show that $\phi(D_A(a, \mathcal{V}))$ is a dense subset of $D_A(\phi, \mathcal{V})(a)$. Let $T \in \mathrm{Mix}(A, \mathcal{V})$. Then $(T^*\phi)(a) = \phi(Ta)$. Letting T range through all $\mathrm{Mix}(A, \mathcal{V})$ the left side is dense in $D_A(\phi, \mathcal{V})(a)$ while the right side is dense in $\phi(D_A(a, \mathcal{V}))$. $\square$

We will find it convenient to work with more general unitary mixing operators on A^*. We let $\overline{\mathrm{Mix}}(A^*, \mathcal{V})$ denote the closure of $\mathrm{Mix}(A^*, \mathcal{V})$ in the point-weak* topology on $B(A^*)$ (the bounded linear operators on A^*). If $\mathcal{V} = \mathcal{U}(A^\sim)$ we simply write $\overline{\mathrm{Mix}}(A^*)$. Since a limit in the point-weak* topology of positive contractions is again a positive contraction, all $T \in \overline{\mathrm{Mix}}(A^*, \mathcal{V})$ are positive contractions. Since the unit ball of $B(A^*)$ is compact in the point-weak* topology, $\overline{\mathrm{Mix}}(A^*, \mathcal{V})$ is a compact set in this topology.

Lemma 1.2. *Let $\phi \in A^*$. Then $D_A(\phi, \mathcal{V}) = \{T\phi : T \in \overline{\mathrm{Mix}}(A^*, \mathcal{V})\}$.*

Proof. Clearly, $T\phi \in D_A(\phi, \mathcal{V})$ for all $T \in \overline{\mathrm{Mix}}(A^*, \mathcal{V})$. Suppose that $\psi \in D_A(\phi, \mathcal{V})$. Then $T_i\phi \to \psi$ in the weak* topology for some net of $\mathcal{V}$-mixing operators $(T_i)_i$ on A^*. Passing to a subnet of $(T_i)_i$ convergent in the point-weak* topology we get that $\psi = T\phi$ for some $T \in \overline{\mathrm{Mix}}(A^*, \mathcal{V})$. $\square$

2. Maximally mixed functionals

Let $\phi \in A^*$. If $\psi \in D_A(\phi)$ we say that ψ is more unitarily mixed than ϕ. We say that ϕ is *maximally (unitarily) mixed* if $D_A(\phi)$ is minimal with respect to the order by inclusion in the lattice of weak*-compact Dixmier subsets of A^*. Thus ϕ is maximally mixed if and only if for all $\psi \in D_A(\phi)$ we have $D_A(\psi) = D_A(\phi)$.

It follows from Zorn's lemma that any weak*-compact Dixmier set contains a maximally mixed functional. In particular, $D_A(\phi)$ contains a maximally mixed functional for all $\phi \in A^*$. Note also that (i) the zero functional is maximally mixed, (ii) if ϕ is tracial then $D_A(\phi) = \{\phi\}$ and hence ϕ is maximally mixed, and (iii) if ϕ is maximally mixed and $\lambda \in \mathbb{C}$ then $\lambda\phi$ is maximally mixed.

Theorem 2.1. *Let $\phi \in A^*$ be maximally mixed. Then the self-adjoint and skew-adjoint parts of ϕ are maximally mixed. If ϕ is self-adjoint, then its positive and negative parts are maximally mixed.*

Proof. Let ϕ_{sa} denote the self-adjoint part of ϕ. Let $\psi \in D_A(\phi_{\mathrm{sa}})$. Then $\psi = T\phi_{\mathrm{sa}}$ for some $T \in \overline{\mathrm{Mix}}(A^*)$ (Lemma 1.2). Mixing operators in $\overline{\mathrm{Mix}}(A^*)$ preserve the self-adjoint part. So ψ is the self-adjoint part of $T\phi$. Since ϕ is maximally mixed and $T\phi \in D_A(\phi)$, there exists $S \in \overline{\mathrm{Mix}}(A^*)$ such that $ST\phi = \phi$. Taking self-adjoint parts we get $S\psi = \phi_{\mathrm{sa}}$. Thus, $\phi_{\mathrm{sa}} \in D_A(\psi)$, as desired. The same argument applies to the skew-adjoint part.

Suppose now that ϕ is self-adjoint (and maximally mixed). Let us show first that $(T\phi)_+ = T\phi_+$ and $(T\phi)_- = T\phi_-$ for any $T \in \overline{\mathrm{Mix}}(A^*)$. Observe that $\|\psi\| \leqslant \|\phi\|$ for all $\psi \in D_A(\phi)$. But, since ϕ is maximally mixed, we must have that $\|\psi\| = \|\phi\|$ for all $\psi \in D_A(\phi)$. That is, all the functionals in $D_A(\phi)$ have the same norm. Applying T on both sides of $\phi = \phi_+ - \phi_-$ we get $T\phi = T\phi_+ - T\phi_-$. Then,

$$\|T\phi_+\| + \|T\phi_-\| \leqslant \|\phi_+\| + \|\phi_-\| = \|\phi\| = \|T\phi\|.$$

It follows that $T\phi_+$ and $T\phi_-$ are orthogonal ([6, Lemma 3.2.3]). By the uniqueness of the Jordan decomposition ([6, Theorem 3.2.5]), $(T\phi)_+ = T\phi_+$ and $(T\phi)_- = T\phi_-$.

That ϕ_+ and ϕ_- are maximally mixed is now straightforward. For suppose that $\psi \in D_A(\phi_+)$. By Lemma 1.2, there exists $T \in \overline{\mathrm{Mix}}(A^*)$ such that $\psi = T\phi_+$. Further, since ϕ is maximally mixed, there exists $S \in \overline{\mathrm{Mix}}(A^*)$ such that $ST\phi = \phi$. Then $S\psi = ST\phi_+ = (ST\phi)_+ = \phi_+$. Thus, ϕ_+ is maximally mixed. The same argument shows that ϕ_- is maximally mixed. □

Due in part to the previous theorem, in the sequel our focus will be on the positive maximally mixed functionals. We warn however that it is not true that a self-adjoint functional whose positive and negative parts are maximally mixed is itself maximally mixed: see Example 3.10.

Theorem 2.2. *The set of maximally mixed functionals is a norm-closed subset of A^*.*

Proof. Let $\phi \in A^*$ be in the norm-closure of the set of maximally mixed functionals. Let $\psi \in D_A(\phi)$. By Lemma 1.2, there exists $T \in \overline{\mathrm{Mix}}(A^*)$ such that $\psi = T\phi$. Let $\varepsilon > 0$. Then there exists a maximally mixed $\tilde{\phi}$ such that $\|\phi - \tilde{\phi}\| < \varepsilon$. Since T is a contraction,

$$\|\psi - T\tilde{\phi}\| = \|T\phi - T\tilde{\phi}\| \leqslant \|\phi - \tilde{\phi}\| < \varepsilon.$$

Since $\tilde{\phi}$ is maximally mixed, there exists $S \in \overline{\mathrm{Mix}}(A^*)$ such that $ST\tilde{\phi} = \tilde{\phi}$. Then,

$$\|S\psi - \tilde{\phi}\| = \|S\psi - ST\tilde{\phi}\| \leqslant \|\psi - T\tilde{\phi}\| < \varepsilon.$$

So $\|\phi - S\psi\| < 2\varepsilon$. Since $D_A(\psi)$ is norm-closed, we have $\phi \in D_A(\psi)$ and hence $D_A(\psi) = D_A(\phi)$. Thus, ϕ is maximally mixed. □

We will show in Examples 2.16 and 2.17 that the set of maximally mixed functionals is not always weak*-closed. We do have the following:

Proposition 2.3. *Let A be a unital C^*-algebra and let $\phi \in A^*_+$.*

(i) *Suppose that for every $a \in A_{\mathrm{sa}}$ and $\varepsilon > 0$ there exists a maximally mixed $\phi' \in A^*_+$ such that $\phi' \leqslant \phi$ and $|\phi(a) - \phi'(a)| < \varepsilon$. Then ϕ is maximally mixed.*

(ii) *Suppose that for every $a \in A_{\mathrm{sa}}$ and $\varepsilon > 0$ there exists a maximally mixed $\phi' \in A^*_+$ such that $\phi' \geqslant \phi$ and $|\phi(a) - \phi'(a)| < \varepsilon$. Then ϕ is maximally mixed.*

(iii) *Suppose that $(\phi_i)_i$ is a norm-bounded net of maximally mixed functionals in A^*_+ which is either upward directed or downward directed relative to the order in A^*_+. Then the net is convergent and the limit is maximally mixed.*

Proof. (i) Let $\psi \in D_A(\phi)$ and suppose that $\psi = T\phi$, where $T \in \overline{\mathrm{Mix}}(A^*)$. Suppose, towards a contradiction, that $\phi \notin D_A(\psi)$. Then by the Hahn-Banach theorem there exist $a \in A_{\mathrm{sa}}$, $t \in \mathbb{R}$ and $\varepsilon > 0$ such that $\rho(a) \leqslant t$ for all $\rho \in D_A(\psi)$ but $\phi(a) \geqslant t + \varepsilon$. Replacing a by $a + \|a\|1$ and t by $t + \|a\|\|\phi\|$, we may assume that $a \geqslant 0$.

By hypothesis, there exists a maximally mixed functional $\phi' \in A^*_+$ such that $\phi' \leqslant \phi$ and $\phi'(a) \geqslant t + \varepsilon/2$. Let $\psi' = T\phi'$. Note that, since T is positive, $\psi' \leqslant \psi$. Since ϕ' is maximally mixed, $\phi' \in D_A(\psi')$. Thus, there exists $S \in \overline{\mathrm{Mix}}(A^*)$ such that $S\psi' = \phi'$. Let $\rho = S\psi$. Then $\phi' \leqslant \rho$ and so $\phi'(a) \leqslant \rho(a) \leqslant t$ since $a \geqslant 0$. This contradicts the fact that $\phi'(a) \geqslant t + \varepsilon/2$. Thus $\phi \in D_A(\psi)$ and hence $D_A(\psi) = D_A(\phi)$.

(ii) This is similar to (i).

(iii) The convergence of the net follows from weak*-compactness, monotonicity and the fact that A is the linear span of A_+. The limit is maximally mixed by (i) and (ii). □

Next we prepare to examine the relation of the maximally mixed functionals of A with those of its ideals and quotients. Theorem 2.6 will tell us that, given an ideal J of A, maximal mixedness of a functional can be read off by its decomposition with respect to A/J and J. Part (i) of the following proposition is a classical key result used to prove permanence of the Dixmier property under suitable extensions; we use part (ii) in an analogous way to handle Dixmier sets of functionals.

Proposition 2.4. *Let $a \in A$ and $\phi \in A^*$. The following are true:*
(i) $D_A(a)$ is equal to the norm-closure of $\mathrm{co}\{e^{ih}ae^{-ih} : h \in A_{\mathrm{sa}}\}$.
(ii) $D_A(\phi)$ is equal to the weak-closure of $\mathrm{co}\{e^{ih}\phi e^{-ih} : h \in A_{\mathrm{sa}}\}$.*

Proof. (i) [FLAG1] For unital A, the result is given in [2, Proposition 2.4]. For non-unital A, we apply this result to $A^\sim$ and use the fact that if $h \in A_{\mathrm{sa}}$ and $t \in \mathbb{R}$ then $e^{i(h+t1)} = e^{it1}e^{ih}$.

(ii) This follows from (i) and the Hahn-Banach theorem. Indeed, if (ii) fails to hold then there is a unitary conjugate of ϕ which does not belong to the weak*-closure of $\mathrm{co}\{e^{ih}\phi e^{-ih} : h \in A_{\mathrm{sa}}\}$. Since A^* with the weak*-topology has dual space A, it follows by the Hahn-Banach separation theorem that there exists $u \in \mathcal{U}(A)$, $a \in A$ and $t \in \mathbb{R}$ such that $\mathrm{Re}(\phi(uau^*)) > t$ and $\mathrm{Re}(\phi(e^{ih}ae^{-ih})) \leqslant t$ for all $h \in A_{\mathrm{sa}}$. It follows from the last inequality and part (i) that $\mathrm{Re}(\phi(x)) \leqslant t$ for all $x \in D_A(a)$. This contradicts the fact that $\mathrm{Re}(\phi(uau^*)) > t$. □

Proposition 2.5. *Let J be a proper, closed two-sided ideal of a unital C^*-algebra A. Let $\iota_J : J \to A$ and $q_J : A \to A/J$ denote the inclusion and quotient maps.*

(i) *The adjoint map $\iota_J^* : A^* \to J^*$ maps $D_A(\phi)$ onto $D_J(\phi|_J)$ for all $\phi \in A^*_+$.*

(ii) *We have* $D_A(\phi) = D_A(\phi, \mathcal{U}(J + \mathbb{C}1))$ *for all* $\phi \in A_+^*$ *such that* $\|\phi\| = \|\phi|_J\|$.

(iii) *The adjoint map* $q_J^*\colon (A/J)^* \to A^*$ *maps* $D_{A/J}(\phi)$ *bijectively to* $D_A(\phi \circ q_J)$ *for all* $\phi \in (A/J)_+^*$.

Proof. If the ideal J is a unital C*-algebra then $A \cong J \oplus A/J$ and all three results (i)-(iii) have a straightforward proof. We thus assume that J is non-unital. Note then that $J + \mathbb{C}1$ may be regarded as the unitization of J.

(i) Let us first show that $\rho \overset{\iota_J^*}{\longmapsto} \rho|_J$ maps $D_A(\phi)$ into $D_J(\phi|_J)$. Let $\psi \in D_A(\phi)$ and suppose that $\psi|_J \notin D_J(\phi|_J)$. Then, by the Hahn-Banach theorem, there exist $a \in J_{\mathrm{sa}}$ and $t \in \mathbb{R}$ such that $\psi(a) > t$ and $\rho(a) \leqslant t$ for all $\rho \in D_J(\phi|_J)$. It follows from Lemma 1.1 applied to $\phi|_J$ and a that $\phi(b) \leqslant t$ for all $b \in D_J(a)$. But, by [2, Remark 2.6], $D_J(a) = D_A(a)$ (since $a \in J$). Hence $\phi(b) \leqslant t$ for all $b \in D_A(a)$. Lemma 1.1, applied now to ϕ and a, implies that $\rho(a) \leqslant t$ for all $\rho \in D_A(\phi)$. Since $\psi \in D_A(\phi)$, we obtain that $\psi(a) \leqslant t$ which gives a contradiction. Thus ι_J^* maps $D_A(\phi)$ into $D_J(\phi|_J)$.

Let us prove surjectivity. Since ι_J^* is weak*-continuous, the image of $D_A(\phi)$ is a weak*-compact convex subset of $D_J(\phi|_J)$. For every $T \in \mathrm{Mix}(A, \mathcal{U}(J + \mathbb{C}1))$ we have $(\phi \circ T)|_J = \phi|_J \circ T|_J$. Clearly, every mixing operator in $\mathrm{Mix}(J)$ has the form $T|_J$ for some $T \in \mathrm{Mix}(A, \mathcal{U}(J + \mathbb{C}1))$. Thus, letting T range through $\mathrm{Mix}(A, \mathcal{U}(J + \mathbb{C}1))$ the functionals $\phi|_J \circ T|_J$ range through a dense subset of $D_J(\phi|_J)$. This shows that the image of $D_A(\phi)$ by ι_J^* is also dense in $D_J(\phi|_J)$.

(ii) Clearly $D_A(\phi, \mathcal{U}(J + \mathbb{C}1)) \subseteq D_A(\phi)$. To prove the opposite inclusion it suffices to show that $u\phi u^* \in D_A(\phi, \mathcal{U}(J + \mathbb{C}1))$ for all $u \in \mathcal{U}(A)$. Let $u \in \mathcal{U}(A)$ and set $\psi = u\phi u^*$. By (i), $\psi|_J \in D_J(\phi|_J)$, so there exists a net of mixing operators $(T_i)_i$ in $\mathrm{Mix}(A, \mathcal{U}(J + \mathbb{C}1))$ such that

$$(\phi \circ T_i)|_J = (\phi|_J) \circ (T_i|_J) \overset{\text{weak}^*}{\longrightarrow} \psi|_J.$$

Passing to a subnet if necessary, we may assume that $\phi \circ T_i \to \psi' \in D_A(\phi, \mathcal{U}(J + \mathbb{C}1))$. Then $\psi'|_J = \psi|_J$. Moreover [FLAG4], $\|\psi'\| \leqslant \|\phi\| = \|\phi|_J\| = \|\psi|_J\|$. By the uniqueness of the norm-preserving extension of a positive functional, we get that $\psi' = \psi$. Thus, $\psi \in D_A(\phi, \mathcal{U}(J + \mathbb{C}1))$.

(iii) The image of $D_{A/J}(\phi)$ by q_J^* is the set $\{\rho \circ q_J : \rho \in D_{A/J}(\phi)\}$. This set is convex, weak*-compact, and contains $\phi \circ q_J$. Moreover, for $u \in \mathcal{U}(A)$ and $\rho \in D_{A/J}(\phi)$ we have $u(\rho \circ q_J)u^* = (v\rho v^*) \circ q_J$, where $v = q_J(u) \in \mathcal{U}(A/J)$. Hence $\{\rho \circ q_J : \rho \in D_{A/J}(\phi)\}$ is invariant under unitary conjugations. It follows that

$$D_A(\phi \circ q_J) \subseteq \{\rho \circ q_J : \rho \in D_{A/J}(\phi)\}.$$

To prove the reverse inclusion it suffices to show that the left side is dense in the right side (since the left side is weak*-compact). By Proposition 2.4 (ii) (applied in A/J), it suffices to show that $e^{ik}\phi e^{-ik} \circ q_J$ belongs to $D_A(\phi \circ q_J)$ for all $k \in (A/J)_{\mathrm{sa}}$. But if $k \in (A/J)_{\mathrm{sa}}$ then we may find $h \in A_{\mathrm{sa}}$ such that $q_J(h) = k$, from which it follows that

$$(e^{ik}\phi e^{-ik}) \circ q_J = e^{ih}(\phi \circ q_J)e^{-ih} \in D_A(\phi \circ q_J),$$

as desired.

We have shown that q_J^* maps $D_{A/J}(\phi)$ onto $D_A(\phi \circ q_J)$. Since q_J^* is also injective, the result follows. $\square$

Let $J \subseteq A$ be as above a proper closed two-sided ideal of A. Let $(A^*_+)^J$ denote the set of functionals $\phi \in A^*$ such that $\|\phi\| = \|\phi|_J\|$. Let $(A^*_+)_J$ denote the functionals $\phi \in A^*$ such that $\phi(J) = \{0\}$. Recall then that every $\phi \in A^*_+$ can be expressed in the form $\phi = \phi_1 + \phi_2$, with $\phi_1 \in (A^*_+)^J$ and $\phi_2 \in (A^*_+)_J$ and that this decomposition is unique (see, for example, [4, 2.11.7]).

Theorem 2.6. *Let A be a unital C^*-algebra and let J be a proper closed ideal of A. Let $\phi \in A^*_+$ and write $\phi = \phi_1 + \phi_2$, where $\phi_1, \phi_2 \in A^*_+$ are such that $\phi_1 \in (A^*_+)^J$ and $\phi_2 \in (A^*_+)_J$.*

(i) *ϕ_1 is maximally mixed if and only if $\phi_1|_J \in J^*_+$ is maximally mixed.*
(ii) *ϕ_2 is maximally mixed if and only if the functional that it induces on A/J is maximally mixed.*
(iii) *ϕ is maximally mixed if and only if both ϕ_1 and ϕ_2 are maximally mixed. Moreover, in this case $D_A(\phi) = D_A(\phi_1) + D_A(\phi_2)$.*

Proof. (i) Suppose first that ϕ_1 is maximally mixed. Let $\psi' \in D_J(\phi_1|_J)$. By Proposition 2.5 (i), there exists $\psi \in D_A(\phi_1)$ such that $\psi|_J = \psi'$. Since ϕ_1 is maximally mixed, $\phi_1 \in D_A(\psi)$. Then, again by Proposition 2.5 (i), $\phi_1|_J \in D_J(\psi')$. Thus, $\phi_1|_J$ is maximally mixed.

Let us prove the converse. Let $\psi \in D_A(\phi_1)$. Then $\psi|_J \in D_J(\phi_1|_J)$ by Proposition 2.5 (i). Since $\phi_1|_J$ is maximally mixed, $\phi_1|_J \in D_J(\psi|_J)$. By Proposition 2.5 (i), there exists $\phi_1' \in D_A(\psi)$ such that $\phi_1'|_J = \phi_1|_J$. Moreover, $\|\phi_1'\| \leqslant \|\psi\| \leqslant \|\phi_1\|$. By the uniqueness of the norm-preserving extension of a positive functional, $\phi_1' = \phi_1$. So $\phi_1 \in D_A(\psi)$, as desired.

(ii) This is a rather straightforward consequence of Proposition 2.5 (iii). Let $\tilde{\phi} \in (A/J)^*$ be such that $\phi = \tilde{\phi} \circ q_J$. Suppose that $\tilde{\phi}$ is maximally mixed. By Proposition 2.5 (iii), if $\psi \in D_A(\phi)$ then $\psi = \tilde{\psi} \circ q_J$ for some $\tilde{\psi} \in D_{A/J}(\tilde{\phi})$. Since $\tilde{\phi}$ is maximally mixed, $\tilde{\phi} \in D_{A/J}(\tilde{\psi})$. Again by Proposition 2.5 (iii), $\phi \in D_A(\psi)$ as desired. Suppose on the other hand that ϕ is maximally mixed. Let $\tilde{\psi} \in D_{A/J}(\tilde{\phi})$. Then $\tilde{\psi} \circ q_J \in D_A(\phi)$. Hence, $\phi \in D_A(\tilde{\psi} \circ q_J)$. By Proposition 2.5 (iii), $\tilde{\phi} \in D_{A/J}(\tilde{\psi})$ as desired.

(iii) Suppose that ϕ is maximally mixed. Let $T \in \overline{\mathrm{Mix}}(A^*)$. Let us show first that $T\phi_1 \in (A^*_+)^J$ and $T\phi_2 \in (A^*_+)_J$. It is clear that $T\phi_2 \in (A^*_+)_J$, since $\phi_2 \in (A^*_+)_J$ and $(A^*_+)_J$ is a Dixmier set. Thus, restricting to J in $T\phi = T\phi_1 + T\phi_2$ we obtain that $(T\phi)|_J = (T\phi_1)|_J$. Since ϕ is maximally mixed, $\phi \in D_A(T\phi)$, and therefore $\phi|_J \in D_J((T\phi)|_J)$ by Proposition 2.5 (i). Hence,

$$\|\phi_1\| = \|\phi|_J\| \leqslant \|(T\phi)|_J\| = \|(T\phi_1)|_J\|.$$

So $\|T\phi_1\| \leqslant \|\phi_1\| \leqslant \|(T\phi_1)|_J\|$, which shows that $T\phi_1 \in (A^*_+)^J$ (by the definition of $(A^*_+)^J$).

To prove that ϕ_1 and ϕ_2 are maximally mixed we proceed as follows: Since ϕ is maximally mixed, there exists $S \in \overline{\mathrm{Mix}}(A^*)$ such that $ST\phi = \phi$. We thus have that $\phi = ST\phi_1 + ST\phi_2$. Using the last paragraph with ST in place of T, we have that $ST\phi_2 \in (A^*_+)_J$ and $ST\phi_1 \in (A^*_+)^J$. By the uniqueness of the decomposition of ϕ into a functional in $(A^*_+)^J$ and one in $(A^*_+)_J$ we conclude that $ST\phi_1 = \phi_1$ and $ST\phi_2 = \phi_2$.

Thus, for any $T \in \overline{\mathrm{Mix}}(A^*)$ there exists $S \in \overline{\mathrm{Mix}}(A^*)$ such that $ST\phi_1 = \phi_1$ and $ST\phi_2 = \phi_2$. In view of Lemma 1.2, this shows that ϕ_1 and ϕ_2 are maximally mixed.

Suppose now that both ϕ_1 and ϕ_2 are maximally mixed. Let us show first that $D_A(\phi) = D_A(\phi_1) + D_A(\phi_2)$. The inclusion $D_A(\phi) \subseteq D_A(\phi_1) + D_A(\phi_2)$ is clear, for if $T \in \overline{\mathrm{Mix}}(A^*)$ then $T\phi = T\phi_1 + T\phi_2$, which belongs to $D_A(\phi_1) + D_A(\phi_2)$, and by Lemma 1.2 $T\phi$ ranges through all of $D_A(\phi)$. Let $\phi_1' \in D_A(\phi_1)$ and $\phi_2' \in D_A(\phi_2)$ and let us show that $\phi_1' + \phi_2' \in D_A(\phi)$. Choose $T \in \overline{\mathrm{Mix}}(A^*)$ such that $T\phi_2 = \phi_2'$, so that $T\phi = T\phi_1 + \phi_2'$. Recall that, as shown above, operators in $\overline{\mathrm{Mix}}(A^*)$ preserve the decomposition of a maximally mixed functional into functionals in $(A^*_+)^J$ and $(A^*_+)_J$. Hence, $T\phi_1 \in (A^*_+)^J$. Since $\phi_1' \in D_A(T\phi_1)$, there exists $S \in \overline{\mathrm{Mix}}(A^*)$ such that $ST\phi_1 = \phi_1'$. Moreover, by Proposition 2.5 (ii), we can choose $S \in \overline{\mathrm{Mix}}(A^*, \mathcal{U}(J + \mathbb{C}1))$. Observe then that $S\phi_2' = \phi_2'$ (since ϕ_2' vanishes on J). Hence, $ST\phi = \phi_1' + \phi_2'$, as desired.

Continue to assume that ϕ_1 and ϕ_2 are maximally mixed and let us show that ϕ is maximally mixed. Let $\phi' \in D_A(\phi)$. Then $\phi' = \phi_1' + \phi_2'$, where $\phi_1' \in D_A(\phi_1)$ and $\phi_2' \in D_A(\phi_2)$. So

$$D_A(\phi) = D_A(\phi_1) + D_A(\phi_2) = D_A(\phi_1') + D_A(\phi_2') = D_A(\phi'),$$

where we use the fact that ϕ_1' and ϕ_2' are maximally mixed, and the result of the previous paragraph, for the final equality. Hence, ϕ is maximally mixed. □

Corollary 2.7. *Let A be a non-unital C*-algebra and $\phi \in A^*_+$. Then ϕ is maximally mixed if and only if its norm preserving extension to $A^\sim$ is maximally mixed.*

In view of the previous corollary in the sequel we focus our attention on unital C*-algebras. Further, since the scalar multiplies of a maximally mixed functional are maximally mixed, we work with states. We denote by $S(A)$ the state space of A and by $S_\infty(A)$ the set of maximally mixed states of A.

Let A be a unital C*-algebra. Consider states $\phi \in S(A)$ of the following two types:

(A) ϕ is tracial,

(B) ϕ factors through a simple quotient A/M without bounded traces.

We will also, where appropriate, speak of type (B) positive functionals (these are positive scalar multiples of type (B) states). Not much effort is needed to see that the states of these types are maximally mixed (for tracial states, this is obvious, whereas for type (B) states, it follows from a short argument in Lemma 2.8 below); this prompts us to ponder whether all maximally mixed states can be described in terms of these ones. We show in Theorem 2.10 that we are close to getting all maximally mixed states by taking the convex hull of these ones – although we don't know whether the set of maximally mixed states is convex, see Question 2.13 below!

Lemma 2.8. *If B is a simple unital C*-algebra with no bounded traces, then for every state $\phi \in S(B)$, $D_B(\phi) = S(B)$, and thus every state on B is maximally mixed. Therefore every type (B) state on a unital C*-algebra is maximally mixed.*

Proof. Suppose for a contradiction that there exists $\psi \in S(B) \setminus D_B(\phi)$. By the Hahn-Banach theorem, there exist $a \in A_{\mathrm{sa}}$ and $t \in \mathbb{R}$ such that $D_A(\phi)(a) \leqslant t$ (i.e., $s \leqslant t$ for all $s \in D_A(\phi)(a)$) and $\psi(a) > t$. Translating by a scalar, we may assume that a is positive.

We then know that $\phi(D_A(a)) \leqslant t$ (Lemma 1.1) and $\psi(a) > t$. But $\|a\| \cdot 1 \in D_A(a)$ (by [5, Théorème 4]), and so $\|a\| \leqslant t$, which contradicts that $\psi(a) > t$.

The final statement now follows by Theorem 2.6 (ii). □

Proposition 2.9. *Let A be a unital C^*-algebra. Let $\phi \in A^*_+$ be maximally mixed and let $\psi \in A^*_+$ be either tracial or type (B). Then $\phi + \psi$ is maximally mixed and $D_A(\phi + \psi) = D_A(\phi) + D_A(\psi)$.*

Proof. If ψ is tracial then $D_A(\phi + \psi) = D_A(\phi) + \psi$, from which the result follows at once. Suppose then that ψ is type (B), i.e., it factors through a simple quotient A/M without bounded traces. Let $\phi = \phi_1 + \phi_2$, where $\phi_1 \in (A^*_+)^M$ and $\phi_2 \in (A^*_+)_M$. Then

$$\phi + \psi = \phi_1 + (\phi_2 + \psi).$$

By Theorem 2.6 (iii), ϕ_1 is maximally mixed. On the other hand, $\phi_2 + \psi$ is type (B) (it factors through A/M), so by Lemma 2.8, it is maximally mixed. Hence, by Theorem 2.6 (iii), $\phi = \phi_1 + (\phi_2 + \psi)$ is maximally mixed. Moreover, Theorem 2.6 (iii) also shows that $D_A(\phi) = D_A(\phi_1) + D_A(\phi_2 + \psi)$. But $D_A(\phi_2 + \psi) = (\phi_2(1) + \psi(1))S(A)_M$, where $S(A)_M = S(A/M) \circ q_M$ (i.e., all states that factor through A/M). So

$$\begin{aligned} D_A(\phi) &= D_A(\phi_1) + \phi_2(1)S(A)_M + \psi(1)S(A)_M \\ &= D_A(\phi_1) + D_A(\phi_2) + D_A(\psi) \\ &= D_A(\phi) + D_A(\psi), \end{aligned}$$

using Theorem 2.6 (iii) again for the last equality. □

Theorem 2.10. *Let A be a unital C^*-algebra, and let Λ denote the convex hull of the set of states that are either tracial or type (B). Then $\overline{\Lambda}^{\|\cdot\|} \subseteq S_\infty(A) \subseteq \overline{\Lambda}^{\mathrm{weak}^*}$.*

Examples 2.16, 2.17, and 3.9 show that neither inequality in the above theorem can be turned into an equality.

Proof. It follows by Proposition 2.9 that $\Lambda \subseteq S_\infty(A)$, and so by Theorem 2.2

$$\overline{\Lambda}^{\|\cdot\|} \subseteq S_\infty(A).$$

On the other hand, to show that $S_\infty(A)$ is contained in the weak*-closure of Λ, it suffices to show that for any $\phi \in S(A)$ the Dixmier set $D_A(\phi)$ has nonempty intersection with $\overline{\Lambda}^{\mathrm{weak}^*}$. Suppose, for the sake of contradiction, that this is not the case for some $\phi \in S(A)$. Then, by the Hahn-Banach theorem, there exists a self-adjoint element a and real numbers $t_1 < t_2$ such that $\psi(a) \leqslant t_1$ for all $\psi \in \Lambda$ and $\psi'(a) \geqslant t_2$ for all $\psi' \in D_A(\phi)$. Translating a by a multiple of the unit we can assume that it is positive. Since $D_A(\phi)(a) = \overline{\phi(D_A(a))}$ (Lemma 1.1), we have that $\phi(a') \geqslant t_2$ for all $a' \in D_A(a)$. On the other hand, $\psi(a) \leqslant t_1$ for every tracial state and every state that factors through a simple quotient without bounded traces. By [3, Theorem 4.12], the distance from $D_A(a)$ to 0 is at most t_1. Thus, there exists $a' \in D_A(a)$ such that $\|a'\| < t_2$. This contradicts that $\phi(a') \geqslant t_2$. □

Corollary 2.11. *Let A be a unital C^*-algebra such that every simple quotient of A has a bounded trace. Then all the maximally mixed states of A are tracial.*

In the case of simple C*-algebras we obtain a complete description of the maximally mixed positive functionals:

Corollary 2.12. *Let A be a simple C*-algebra.*

(i) *If A is unital and has at least one non-zero bounded trace then every maximally mixed positive functional on A is tracial.*
(ii) *If A is unital and has no bounded traces then all the positive functionals on A are maximally mixed.*
(iii) *If A is non-unital then every maximally mixed positive functional on A is tracial.*

Proof. (i) follows from Corollary 2.11, while (ii) is Lemma 2.8. For (iii), note that $A^{\sim}$ has only one simple quotient, namely $\mathbb{C}$, and it has a bounded trace. Hence by Corollary 2.11, every maximally mixed state on $A^{\sim}$ is tracial, and then (iii) follows from Theorem 2.6 (i). □

Question 2.13. Let A be a unital C*-algebra. Is the set $S_\infty(A)$ of maximally mixed states convex?

A closely related question is the following:

Question 2.14. Do we have $D_A(\phi + \psi) = D_A(\phi) + D_A(\psi)$ for all $\phi, \psi \in S_\infty(A)$?

An affirmative answer to this question also answers affirmatively Question 2.13. Indeed, suppose that Question 2.14 has an affirmative answer and say we are given $\phi, \psi \in S_\infty(A)$ and $\phi' \in D_A(\phi)$ and $\psi' \in D_A(\psi)$. Then

$$D_A(\phi + \psi) = D_A(\phi) + D_A(\psi) = D_A(\phi') + D_A(\psi') = D_A(\phi' + \psi').$$

Recall that Proposition 2.9 answers Question 2.14 affirmatively in the case that ψ is either tracial or type (B).

Turning to the question of whether the containment $S_\infty(A) \subseteq \overline{\Lambda}^{\text{weak}^*}$ is strict, where Λ is as defined in Theorem 2.10, it is evident from that theorem that (non-)strictness of this inequality is equivalent to the natural question of whether $S_\infty(A)$ is weak*-closed. The next proposition gives an obstruction to $S_\infty(A)$ being weak*-closed – in fact, it is the only obstruction we have been able to find, see Question 2.18.

Proposition 2.15. *Let A be a unital C*-algebra such that $S_\infty(A)$ is a weak*-closed subset of $S(A)$. Then the set of all maximal ideals M such that A/M is either isomorphic to $\mathbb{C}$ or has no bounded traces is a closed subset of* $\mathrm{Prim}(A)$.

Proof. Let X denote the set of all all maximal ideals M such that A/M is either isomorphic to $\mathbb{C}$ or has no bounded traces. Let $J = \bigcap_{M \in X} M$. Let $N \in \mathrm{Prim}(A)$ be an adherence point of X, i.e, $J \subseteq N$. Then every state on A that factors through A/N is a weak* limit of convex combinations of states that factor through A/M, with $M \in X$ ([4]). Notice that $S_\infty(A/M) = S(A/M)$ for all $M \in X$. Thus, all the states that factor through A/M, with $M \in X$, are maximally mixed. It follows that all states factoring through A/N are maximally mixed, and so all states of A/N are maximally mixed by Theorem 2.6 (ii).

Since N is primitive, let $\phi \in S(A/N)$ be a pure state whose GNS representation is faithful. Then any pure state ψ on A/N is a weak* limit of vector states (with respect to the GNS representation) by [4, Corollary 3.4.3]. By the unitary version of Kadison's Transitivity Theorem ([4, Theorem 2.8.3 (iii)]), each of these vector states is in fact unitarily equivalent to ϕ, and thus ψ is a weak* limit of unitary conjugates of ϕ. By approximating arbitrary states on A/N by convex combinations of pure states, we find that $S(A/N) = D_{A/N}(\phi)$. Since all states on A/N are maximally mixed, it follows that $S(A/N) = D_{A/N}(\psi)$ for all $\psi \in S(A/N)$.

This implies that A/N is simple, for otherwise the states factoring through a non-trivial quotient would form a proper Dixmier set. From Corollary 2.12 we see that A/N must be either isomorphic to $\mathbb{C}$ or without bounded traces. Thus, $N \in X$. □

The examples below show that $S_\infty(A)$ may fail to be weak*-closed.

Example 2.16. Fix a simple unital C*-algebra B without bounded traces (e.g., the Cuntz algebra $\mathcal{O}_2$). Let A be the C*-subalgebra of $C([0,1], M_2(B))$ of functions f such that $f(1) \in M_2(\mathbb{C}) \subseteq M_2(B)$. For each $t \in [0,1]$ let $M_t = \{f \in A : f(t) = 0\}$. Then $A/M_t \cong M_2(B)$ for all $0 \leqslant t < 1$. So M_t is a maximal ideal such that A/M_t is simple without bounded traces. The maximal ideal M_1 is an adherence point of the set $\{M_t : 0 \leqslant t < 1\}$. However, $A/M_1 \cong M_2(\mathbb{C})$ has a bounded trace and is not isomorphic to $\mathbb{C}$. Thus, $S_\infty(A)$ is not weak*-closed, by Proposition 2.15.

Example 2.17. Again fix a simple unital C*-algebra B without bounded traces. Let A be the C*-subalgebra of $C(\{1,2,\ldots,\infty\}, (B\otimes\mathcal{K})^\sim)$ of f such that $f(n) \in M_n(B) + \mathbb{C}1$ for all $n \in \mathbb{N}$, where we regard $M_n(B)$ embedded in $B \otimes \mathcal{K}$ as the top right corner. For each $n \in \mathbb{N}$ define $I_n = \{f \in A : e_n f(n) = 0\}$, where e_n is the unit of $M_n(B)$. Then I_n is a maximal ideal for all $n = 1, 2, \ldots$ and $A/I_n \cong M_n(B)$ has no bounded traces. Since $\bigcap_n I_n = \{0\}$, the set $\{I_n : n \in \mathbb{N}\}$ is dense in $\mathrm{Prim}(A)$. Consider the ideal $I_\infty = \{f : f(\infty) = 0\}$. Since $A/I_\infty = (B\otimes\mathcal{K})^\sim$ is a prime C*-algebra, $I_\infty \in \mathrm{Prim}(A)$. But I_∞ is not maximal. By Proposition 2.15, $S_\infty(A)$ is not weak*-closed.

If one wanted an algebra A with no bounded traces in which $S_\infty(A)$ is not weak*-closed, one can simply tensor the example just given with a nuclear, unital, simple, traceless C*-algebra (this operation does not change the ideal lattice, so the same obstruction applies).

Question 2.18. Is the converse of Proposition 2.15 true? That is, let A be separable and unital. Suppose that the set of maximal ideals M such that A/M is either isomorphic to $\mathbb{C}$ or has no bounded traces is a closed subset of $\mathrm{Prim}(A)$. Is $S_\infty(A)$ weak*-closed?

In the next section we answer affirmatively Questions 2.13, 2.14, and 2.18 for C*-algebras with the Dixmier property.

3. C*-algebras with the Dixmier property

Let A be a unital C*-algebra. Let $Z(A)$ denote its center. Recall that A is said to have the Dixmier property if $D_A(a) \cap Z(A)$ is non-empty for all $a \in A$. Henceforth in this section we assume that A is a unital C*-algebra with the Dixmier property.

To analyze the maximally mixed states for such A, we will make frequent use of a description of $D_A(a) \cap Z(A)$ (for a self-adjoint) found in [3] (between Theorem 2.6 and Corollary 2.7, with details in the proof of Theorem 2.6). Let $\hat{Z}$ denote the spectrum of

$Z(A)$, which, by weak centrality of A, we identify with the set of maximal ideals of A. For $a \in A$ self-adjoint, define $f_a, g_a : \hat{Z} \to \mathbb{R}$ by

$$f_a(M) := \begin{cases} \min \mathrm{sp}(q_M(a)), & \text{if } A/M \text{ has no bounded traces;} \\ \tau_M(a), & \text{otherwise,} \end{cases}$$

where τ_M is the (necessarily unique) tracial state on A which factors through A/M. Likewise,

$$g_a(M) := \begin{cases} \max \mathrm{sp}(q_M(a)), & \text{if } A/M \text{ has no bounded traces;} \\ \tau_M(a), & \text{otherwise.} \end{cases} \tag{3.1}$$

Then f_a is upper semicontinuous, g_a is lower semicontinuous, $f_a \leqslant g_a$, and, identifying $Z(A) = C(\hat{Z})$ now,

$$D_A(a) \cap Z(A) = \{z \in C(\hat{Z}) : z = z^* \text{ and } f_a \leqslant z \leqslant g_a\}.$$

Let us say that two maximally mixed bounded functionals ϕ and ψ are equivalent if they generate the same Dixmier set, i.e., $D_A(\phi) = D_A(\psi)$.

Proposition 3.1. *The equivalence classes of maximally mixed, bounded functionals on A are in bijective correspondence with the bounded functionals on the center of A. The correspondence is given by the restriction map $\phi \mapsto \phi|_{Z(A)}$, for ϕ maximally mixed.*

Proof. Taking self-adjoint and skew-adjoint parts, we may reduce to the case of self-adjoint functionals. Any two equivalent self-adjoint functionals agree on the center, so the mapping is well defined on equivalence classes. To see that it is onto, fix a self-adjoint functional μ on the center. Then the set of all $\phi \in A^*_{sa}$ whose restriction to $Z(A)$ is μ is a weak*-compact Dixmier set. It thus must contain maximally mixed functionals.

Let us now show that the mapping is injective. Let $\phi, \psi A^*_{sa}$ be two maximally mixed self-adjoint functionals that agree on the center. Suppose for a contradiction that $D_A(\phi) \neq D_A(\psi)$. Then $D_A(\phi)$ and $D_A(\psi)$ are disjoint. By the Hahn-Banach theorem, we can find $a \in A_{\mathrm{sa}}$ and real numbers $t_1 < t_2$ such that $\phi'(a) \leqslant t_1$ for all $\phi' \in D_A(\phi)$ and $\psi'(a) \geqslant t_2$ for all $\psi' \in D_A(\psi)$. By Lemma, $\phi(a') \leqslant t_1$ and $\psi(a') \geqslant t_2$ for all $a' \in D_A(a)$. This holds in particular for $a' \in D_A(a) \cap Z(A)$. This contradicts that ϕ and ψ agree on $Z(A)$. □

Remark 3.2. The previous proposition implies that if A has the Dixmier property then $D_A(\phi)$, for $\phi \in S(A)$, contains a unique equivalence class of maximally mixed states; namely, the maximally mixed states that agree with ϕ on $Z(A)$. This is in general not true for C*-algebras without the Dixmier property. Take for example A to be a simple unital C*-algebra with at least two tracial states and let ϕ be a pure state on A. Then $D_A(\phi)$ is the set of all states, so it contains both tracial states (which are inequivalent maximally mixed states).

We need the following little lemma in the proceeding theorem.

Lemma 3.3. *Let X be a Hausdorff topological space, let μ be a Radon probability measure on X, and let $f : X \to \mathbb{R}$ be a bounded lower semicontinuous function. Then*

$$\int_X f \, d\mu = \sup \int_X g \, d\mu,$$

where the supremum is taken over upper semicontinuous functions $g : X \to \mathbb{R}$ which are (pointwise) dominated by f.

Proof. Without loss of generality, $f \geqslant 0$. We may approximate f uniformly by simple lower semicontinuous functions, i.e., positive scalar linear combinations of characteristic functions of open sets. Thus, it suffices to handle the case that f is the characteristic function of an open set, say $f = \chi_U$.

In this case, since μ is inner regular, $\mu(X)$ is the supremum of measures of compact sets K contained in U, so

$$\begin{aligned}\int_X f \, d\mu &= \mu(X) \\ &= \sup_K \mu(K) \\ &= \sup_K \int_X \chi_K \, d\mu,\end{aligned}$$

where the suprema are taken over compact sets contained in U; but now we are done, since each χ_K is upper semicontinuous. □

Theorem 3.4. *Let A be a unital C*-algebra with the Dixmier property. Let $\phi \in S(A)$. The following are equivalent:*

(i) *ϕ satisfies that*

$$\phi(a) \leqslant \sup\{\phi(z) : z \in D_A(a) \cap Z(A)\} \qquad (a \in A_+). \tag{3.2}$$

(ii) *ϕ is maximally mixed.*

Proof. (i)⇒(ii). Suppose for a contradiction that there exists $\psi \in D_A(\phi)$ such that $\phi \notin D_A(\psi)$. Then there exists a self-adjoint element a and $t \in \mathbb{R}$ separating $D_A(\psi)$ and ϕ. That is, $\psi'(a) \leqslant t$ for all $\psi' \in D_A(\psi)$ and $\phi(a) > t$. Translating a by a scalar multiple of the unit we may assume that it is positive. By Lemma 1.1, we get that $\psi(a') \leqslant t$ for all $a' \in D_A(a)$. From $\psi \in D_A(\phi)$ we deduce that $\psi(a') = \phi(a')$ for all $a' \in Z(A)$. Hence

$$\begin{aligned}\phi(a) &\leqslant \sup\{\phi(a') : a' \in D_A(a) \cap Z(A)\} \\ &= \sup\{\psi(a') : a' \in D_A(a) \cap Z(A)\} \leqslant t.\end{aligned}$$

This contradicts that $\phi(a) > t$.

(ii)⇒(i). First, let us show that if a maximally mixed state ϕ satisfies (3.2) then so do all the states equivalent to it. Let ϕ be a state that satisfies (3.2) and let $\psi \in D_A(\phi)$. Say $\psi = \lim_i \phi \circ T_i$, where $(T_i)_i$ is a net of mixing operators in $\mathrm{Mix}(A)$. Let $a \in A_+$. Since $D_A(T_i a) \subseteq D_A(a)$,

$$\begin{aligned}\phi(T_i a) &\leqslant \sup\{\phi(z) : z \in D_A(T_i a) \cap Z(A)\} \\ &\leqslant \sup\{\phi(z) : z \in D_A(a) \cap Z(A)\}.\end{aligned}$$

Hence

$$\begin{aligned}\psi(a) &= \lim_i \phi(T_i \cdot a) \\ &\leqslant \sup\{\phi(z) : z \in D_A(a) \cap Z(A)\} \\ &= \sup\{\psi(z) : z \in D_A(a) \cap Z(A)\},\end{aligned}$$

where the last equality is valid since ϕ and ψ agree on $Z(A)$.

By Proposition 3.1, it now suffices to show that every probability (Radon) measure μ on the center can be extended to a state on A satisfying (3.2). For each self-adjoint element $a \in A$ let us define $p_\mu(a) \in [0, \infty)$ by

$$p_\mu(a) := \int_{\widehat{Z}} g_a(M)\, d\mu(M),$$

where $g_a \colon \widehat{Z} \to [0, \infty)$ is the lower semicontinuous function on the spectrum of the center associated to a in (3.1). Notice that p_μ is a seminorm and that $p_\mu(a) \leqslant \|a\|$ for all $a \in A_{\mathrm{sa}}$ (since $g_a \leqslant \|a\|$). For any self-adjoint central element z we have that

$$\left| \int z(M)\, d\mu(M) \right| \leqslant \int |z(M)|\, d\mu(M) = p_\mu(z).$$

So we can extend μ by the Hahn-Banach extension theorem to a self-adjoint functional ϕ such that

$$|\phi(a)| \leqslant p_\mu(a) \qquad (a \in A_{\mathrm{sa}}).$$

Notice that $\phi(1) = 1$ and that $\|\phi\| \leqslant 1$, since $p_\mu(a) \leqslant \|a\|$ for all $a \in A_{\mathrm{sa}}$. Hence, ϕ is a state.

Finally, to establish (3.2), we will show that $p_\mu(a)$ is dominated by the right-hand side of (3.2) (though we don't need it, in fact this implies that these two quantities are equal, as the reverse inequality is straightforward). Since g_a is lower semicontinuous, by Lemma 3.3, for any $\varepsilon > 0$, we may find an upper semicontinuous function $w \in C(\hat{Z})$ such that $w \leqslant g_a$ and $\int w(M)\, d\mu(M) > \int g_a(M)\, d\mu(M) - \varepsilon$. By the Katetev-Tong insertion theorem, we may find a continuous function $z_0 \in C(\hat{Z})_+$ such that

$$\begin{matrix} f_a \\ w \end{matrix} \leqslant z_0 \leqslant g_a,$$

and therefore

$$\int z_0(M)\, d\mu(M) \geqslant \int w(M)\, d\mu(M) > \int g_a(M)\, d\mu(M) - \varepsilon = p_\mu(a) - \varepsilon.$$

Thus

$$p_\mu(a) \leqslant \sup\{\phi(z) : z \in D_A(a) \cap Z(A)\},$$

as required. □

Corollary 3.5. *Let A be a unital C*-algebra with the Dixmier property. Then $S_\infty(A)$ is a convex set. Moreover, if $\phi, \psi \in S_\infty(A)$ then $D_A(\phi + \psi) = D_A(\phi) + D_A(\psi)$.*

Proof. To show that $S_\infty(A)$ is convex, we show that the states that satisfy (3.2) form a convex set. Let $\phi, \psi \in S_\infty(A)$. Let $a \in A_+$ and $\varepsilon > 0$. Since ϕ and ψ satisfy (3.2), there exist $x, y \in D_A(a) \cap Z(A)$ such that

$$\phi(a) \leqslant \phi(x) + \varepsilon \text{ and } \psi(a) \leqslant \psi(y) + \varepsilon.$$

By the structure of $D_A(a) \cap Z(A)$ we know that it is a lattice. So we can choose $z \in D_A(a) \cap Z(A)$ such that $x, y \leqslant z$. Now if ρ is a convex combination of ϕ and ψ then $\rho(a) \leqslant \rho(z) + \varepsilon$. This shows that ρ satisfies (3.2) and is therefore maximally mixed.

Let us prove that $D_A(\phi + \psi) = D_A(\phi) + D_A(\psi)$ for all $\phi, \psi \in S_\infty(A)$. The inclusion $D_A(\phi + \psi) \subseteq D_A(\phi) + D_A(\psi)$ is straightforward: if $T \in \overline{\mathrm{Mix}}(A^*)$ then

$$T(\phi + \psi) = T\psi + T\psi \in D_A(\phi) + D_A(\psi),$$

and letting T range through $\overline{\mathrm{Mix}}(A^*)$, $T(\phi+\psi)$ ranges through all of $D_A(\phi+\psi)$ (Lemma 1.2).

Let $\phi, \psi \in S_\infty(A)$ and suppose, for a contradiction, that there exist $\phi' \in D_A(\phi)$ and $\psi' \in D_A(\psi)$ such that $\phi' + \psi' \notin D_A(\phi + \psi)$. Then there exist $a \in A_{\mathrm{sa}}$ and $t \in \mathbb{R}$ such that $\rho(a) \leqslant t$ for all $\rho \in D_A(\phi + \psi)$ while $(\phi' + \psi')(a) > t$. Translating a by a scalar multiple of the unit we may assume that a is positive. By Lemma 1.1, $(\phi + \psi)(b) \leqslant t$ for all $b \in D_A(a)$. Since $\phi + \psi$ and $\phi' + \psi'$ agree on $Z(A)$, we obtain that $(\phi' + \psi')(b) \leqslant t$ for all $b \in D_A(a) \cap Z(A)$. By convexity, $\frac{1}{2}(\phi' + \psi') \in S_\infty(A)$. It follows by Theorem 3.4 that, $(\phi' + \psi')(a) \leqslant t$, which contradicts our choice of a and t. $\square$

Remark 3.6. The C*-algebras in Examples 2.16 and 2.17 both have the Dixmier property (this can be deduced from [3, Theorem 1.1]). So $S_\infty(A)$ may fail to be weak*-closed for C*-algebras with the Dixmier property.

Theorem 3.7. *Let A be a unital C*-algebra with the Dixmier property. The following are equivalent.*

(i) *The set $S_\infty(A)$ is weak*-closed;*
(ii) *The set of maximal ideals M such that A/M is either isomorphic to $\mathbb{C}$ or has no bounded traces is a closed subset of* $\mathrm{Prim}(A)$*;*
(iii) *For each self-adjoint $a \in A$, the set $D_A(a) \cap Z(A)$ contains a maximal element.*

Proof. (i)⇒(ii): This is Proposition 2.15 (no Dixmier property required).

(ii)⇒(iii): By translating, we may assume that $a \geqslant 0$. Let X denote the set of $M \in \mathrm{Max}(A)$ such that A/M is either isomorphic to $\mathbb{C}$ or has no bounded traces, and we assume that this set is closed in $\mathrm{Prim}(A)$. It is evident from the description of $D_A(a) \cap Z(A)$, at the beginning of this section, that we need only show that the function $g_a : \hat{Z} \to \mathbb{R}$ from (3.1) is continuous. Since g_a is always lower semicontinuous, it remains to show that it is upper semicontinuous. Let $t > 0$. Set

$$Y := \{M \in \mathrm{Max}(A) : T(A/M) \neq \varnothing\},$$

which is closed by [3, Theorem 2.6]; for $M \in Y$, A/M has a unique tracial state which we denote τ_M. Then

$$\{M \in Y : \tau_M(a) \geqslant t\}$$

is closed in $\mathrm{Max}(A)$. Also, $\{M \in \mathrm{Prim}(A) : \|q_M(a)\| \geqslant t\}$ is a compact subset of $\mathrm{Prim}(A)$ ([4]), from which (along with that X is closed) we deduce that

$$\{M \in \mathrm{Prim}(A) : \|q_M(a)\| \geqslant t\} \cap X$$

is compact. Since $\mathrm{Max}(A)$ is Hausdorff, the set above is also closed in $\mathrm{Max}(A)$. Therefore,

$$\{M \in Y : \tau_M(a) \geqslant t\} \cup (\{M \in \mathrm{Prim}(A) : \|q_M(a)\| \geqslant t\} \cap X)$$

is closed in $\mathrm{Max}(A)$. But this set is $g_a^{-1}([t,\infty))$, and therefore, g_a is upper semicontinuous.

(iii)⇒(i): For each self-adjoint element $a \in A$, let z_a denote the maximal element of $D_A(a) \cap Z(A)$, which exists since we are assuming (iii). Given a state ϕ, the inequality (3.2) is equivalent to $\phi(a) \leqslant \phi(z_a)$ for all $a \in A_+$. The latter inequality is clearly preserved under weak* limits. By Theorem 3.4, $S_\infty(A)$ is weak*-closed. ☐

We recover as a corollary Alberti's theorem on the maximally mixed states of a von Neumann algebra ([1]):

Corollary 3.8. *Let A be a von Neumann algebra. Then $S_\infty(A)$ agrees with the weak*-closure of the convex hull of the set of tracial states and the type (B) states.*

Proof. One breaks up the algebra into a finite and a properly infinite one and deals with each separately. By a theorem of Halpern, in a properly infinite von Neumann algebra the set of maximal ideals is a closed subset of $\mathrm{Prim}(A)$. ☐

We end this section by taking advantage of the insight we have gained in the case of the Dixmier property, to provide some examples alluded to earlier. The first example shows that the set of maximally mixed states may be larger than the norm-closed convex hull of the tracial states and type (B) states.

Example 3.9. Let B be a simple unital C*-algebra with no bounded traces, and set $A := C([0,1],B)$. If ϕ is in the norm-closed convex hull of the type (B) states, then the state ϕ induces on the centre is in the norm-closed convex hull of point-masses, and therefore corresponds to a discrete measure on $[0,1]$. However, A has the Dixmier property by [3, Theorem 2.6], and by Theorem 3.7, $S_\infty(A)$ is weak*-closed, and therefore, in fact, all of $S(A)$ (since every pure state is of type (B)). So the norm-closed convex hull of the type (B) states (and tracial states, as there are none) is only a small part of $S_\infty(A)$ in this case.

The next example addresses the converse to Theorem 2.1.

Example 3.10. Let A be a simple unital C*-algebra with no bounded traces. Let ϕ be a nonzero functional on A such that $\phi(1) = 0$. Then ϕ is not maximally mixed, because if it were, then since the zero functional is maximally mixed, it would follow by Proposition 3.1 that $D_A(\phi) = D_A(0) = \{0\}$. However, by Corollary 2.12 (ii), both the positive and negative parts of ϕ are maximally mixed.

4. Hausdorff primitive spectrum

Here we impose a different property – Hausdorffness of the primitive ideal space – to make the study of the structure of $S_\infty(A)$ tractable.

Theorem 4.1. *Let A be a unital C*-algebra with Hausdorff primitive spectrum.*

(i) *Suppose that A has no tracial states. Then every state of A is maximally mixed.*
(ii) *Suppose that $T(A) \neq \varnothing$. Then the set*

$$Y := \{M \in \mathrm{Max}(A) : T(A/M) \neq \varnothing\}$$

is non-empty and closed in $\mathrm{Max}(A)$ *and*

$$S_\infty(A) = \mathrm{co}(T(A) \cup S(A)^J),$$

where $J := \bigcap_{M \in Y} M$ *is a proper closed ideal of* A*, and* $S(A)^J$ *consists of all states in* $S(A)$ *which arise as extensions of states in* $S(J)$*.*

(iii) *Questions 2.13, 2.14, 2.18 all have affirmative answers for* A*.*

Proof. Observe first that, since $\mathrm{Prim}(A)$ is Hausdorff, $\mathrm{Prim}(A) = \mathrm{Max}(A) = \mathrm{Glimm}(A)$, and these spaces are all homeomorphic to $\mathrm{Max}(Z(A))$ via the assignment $M \mapsto M \cap Z(A)$. For each maximal ideal N of $Z(A)$, let ϕ_N be the unique pure state of $Z(A)$ with kernel equal to N.

(i) Since the continuous functions on the compact Hausdorff space $\mathrm{Prim}(A)$ separate the points, it follows from the Dauns-Hofmann theorem that A is a central C^*-algebra. Combining this with the fact that $T(A)$ is empty, we obtain from [3, Theorem 2.6] that A has the Dixmier property. Every pure state of A is of type (B), so by Theorem 2.10, $S_\infty(A)$ is weak* dense in $S(A)$. It follows from Theorem 3.7 that every state of A is maximally mixed.

(ii) Since $T(A)$ is non-empty, it contains an extreme point τ by the Krein-Milman theorem. By [3, Lemma 2.4], $\tau|_{Z(A)}$ is a pure state of $Z(A)$ and hence annihilates $M \cap Z(A)$ for some $M \in \mathrm{Max}(A)$. By the Cauchy–Schwartz inequality for states, τ annihilates the Glimm ideal $(M \cap Z(A))A$. But, as noted above, $(M \cap Z(A))A = M$ and so τ induces a tracial state of A/M. Thus Y is non-empty. Moreover, $\tau(J) = \{0\}$ and so, by the Krein-Milman theorem, every tracial state of A annihilates J.

To show that Y is closed, suppose that (M_i) is a net in Y that is convergent to $M \in \mathrm{Max}(A)$. For each i, let τ_i be a tracial state of A that vanishes on M_i. Since $T(A)$ is weak*-compact, there exist $\tau \in T(A)$ and a subnet (τ_{i_j}) such that $\tau_{i_j} \to_j \tau$. Then

$$\tau|_{Z(A)} = \lim_j \phi_{M_{i_j} \cap Z(A)} = \phi_{M \cap Z(A)}.$$

It follows from the Cauchy–Schwartz inequality for states that τ annihilates the Glimm ideal $(M \cap Z(A))A$ and so τ induces a tracial state of A/M as before. Thus $M \in Y$, as required.

Since Y is closed, every maximal ideal of A/J has the form M/J for some $M \in Y$ and hence every simple quotient of A/J has a tracial state. It follows by Corollary 2.11 that $S_\infty(A/J) = T(A/J)$. Letting S_2 be the set of maximally mixed states of $S_\infty(A)$ which factor through A/J, it follows by Theorem 2.6 (ii) that

$$S_2 = S_\infty(A/J) \circ q_J = T(A/J) \circ q_J = T(A).$$

Under the Dauns–Hofmann isomorphism between $Z(A)$ and $C(\mathrm{Prim}(A))$, $Z(J)$ corresponds to $C_0(\mathrm{Prim}(J))$, where $\mathrm{Prim}(J)$ is identified with an open subset of $\mathrm{Prim}(A)$ (namely $\mathrm{Prim}(A) \setminus Y$) in the usual way. It follows that $Z(J)$ separates the primitive ideals of $J + \mathbb{C}1$ and hence $J + \mathbb{C}1$ is a central C^*-algebra. Since J has no tracial states (this follows from [3, Lemma 2.2]), so that $J + \mathbb{C}$ has a unique tracial state, namely the one factoring through the quotient $(J + \mathbb{C}1)/J$. Hence by [3, Theorem 2.6], $J + \mathbb{C}1$ has the Dixmier property. We also have that $\mathrm{Prim}(J + \mathbb{C}1) = \mathrm{Max}(J + \mathbb{C}1)$, with every simple quotient being either traceless or isomorphic to $\mathbb{C}$, and thus by Theorem 3.7,

$S_\infty(J+\mathbb{C}1)$ is weak*-closed. Since every pure state is either type (B) or tracial, it now follows from Theorem 2.10 that $S_\infty(J+\mathbb{C}1) = S(J+\mathbb{C})$. By Theorem 2.6 (i) (used once with $J \lhd J+\mathbb{C}1$ and again with $J \lhd A$) that $S(A)^J \subseteq S_\infty(A)$. Thus, letting S_1 be the set of maximally mixed states of A which are extensions of states from J, we have

$$S_1 = S(A)^J.$$

By Theorem 2.6 (iii), we have

$$S_\infty(A) = \mathrm{co}(S_1 \cup S_2) = \mathrm{co}(S(A)^J \cup T(A)),$$

as required.

(iii) It is evident in both the cases covered by (i) and (ii) that $S_\infty(A)$ is convex and weak*-closed. Now let $\phi, \psi \in S_\infty(A)$, and let's argue that $D_A(\phi+\psi) = D_A(\phi)+D_A(\psi)$. In case (i), we saw that A has the Dixmier property, so this holds by Corollary 3.5.

In case (ii), write $\phi = \phi_1 + \phi_2$ and $\psi = \psi_1 + \psi_2$ where ϕ_1, ψ_1 are positive tracial functionals and ϕ_2, ϕ_2 are non-negative scalar multiples of states in $S(A)^J$; by Theorem 2.6 (i), $\phi_2|_J$ and $\psi_2|_J$ are maximally mixed in $S(A/J)$. Since $J+\mathbb{C}1$ has the Dixmier property (seen in the proof of (ii)), we have by Corollary 3.5 that

$$D_J((\phi_2+\psi_2)|_J) = D_J(\phi_2|_J) + D_J(\psi_2|_J).$$

Then we have

$$\begin{aligned}
D_A(\phi+\psi) &= D_A(\phi_1+\phi_2+\psi_1+\psi_2) \\
&= D_A(\phi_1+\psi_1) + D_A(\phi_2+\psi_2) \\
&= D_A(\phi_1) + D_A(\psi_1) + D_J((\phi_2+\psi_2)|_J) \circ \iota_J^* \\
&= D_A(\phi_1) + D_A(\psi_1) + (D_J(\phi_2|_J) + D_J(\psi_2)) \circ \iota_J^* \\
&= D_A(\phi_1) + D_A(\psi_1) + D_A(\phi_2) + D_A(\psi_2) \\
&= D_A(\phi) + D_A(\psi)
\end{aligned}$$

where we used Proposition 2.5 (i) in the third and fifth equalities, and Proposition 2.9 (the case that one of the states is tracial) in the second, third, and final equalities. □

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