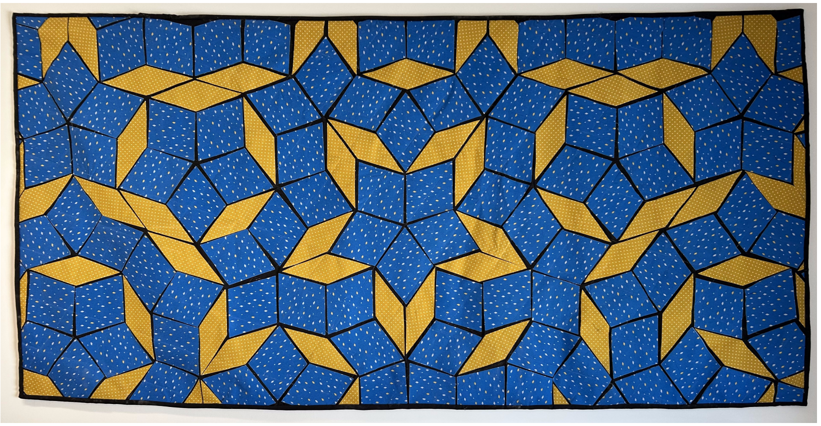
A single tile that fills the plane almost-symmetrically

Aaron Tikuisis



Tiles and tilings

A tile is a nice bounded set in $\mathbb{R}^d$.

(Usually $d = 2$.)

E.g.

"Nice set" means a polygon, or something like a polygon.

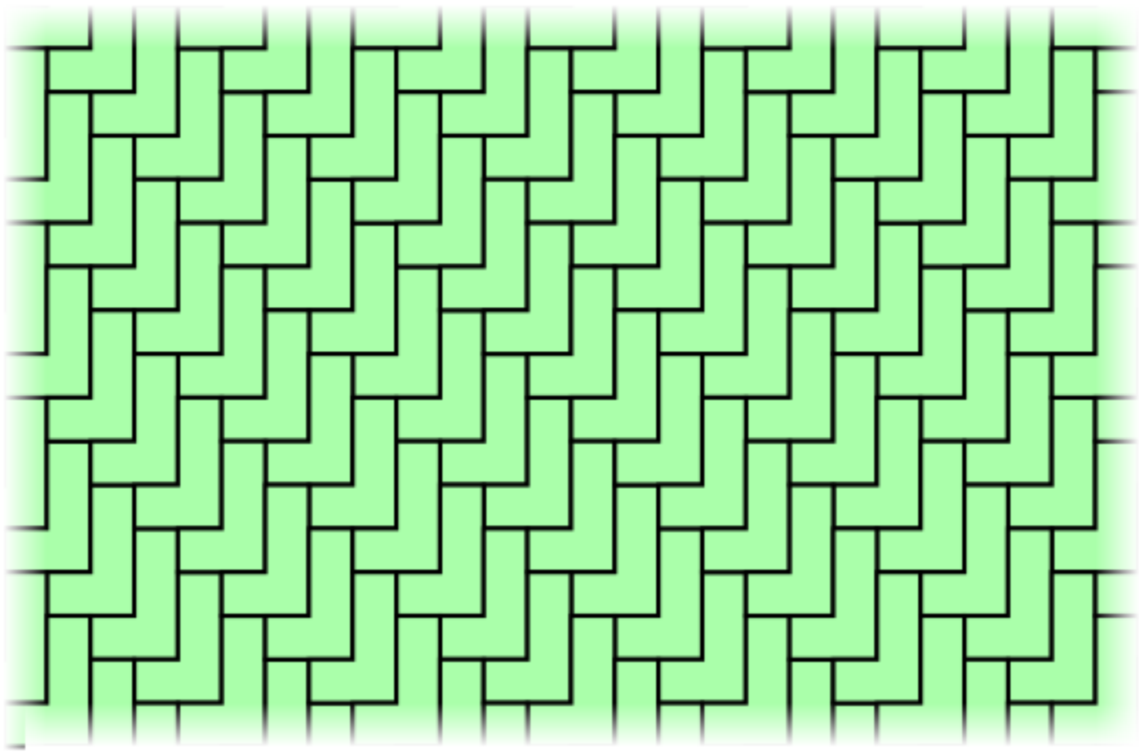
Tiles and tilings

Given a finite set $\{T_1, \ldots, T_k\}$ of tiles, a tiling is a covering of $\mathbb{R}^d$ by copies of the tiles, with no overlap.

E.g.

Tile:

Tiling:



Tiles and tilings

Given a finite set $\{T_1, \ldots, T_k\}$ of tiles, a tiling is a covering of $\mathbb{R}^d$ by copies of the tiles, with no overlap.

Formally:

Definition

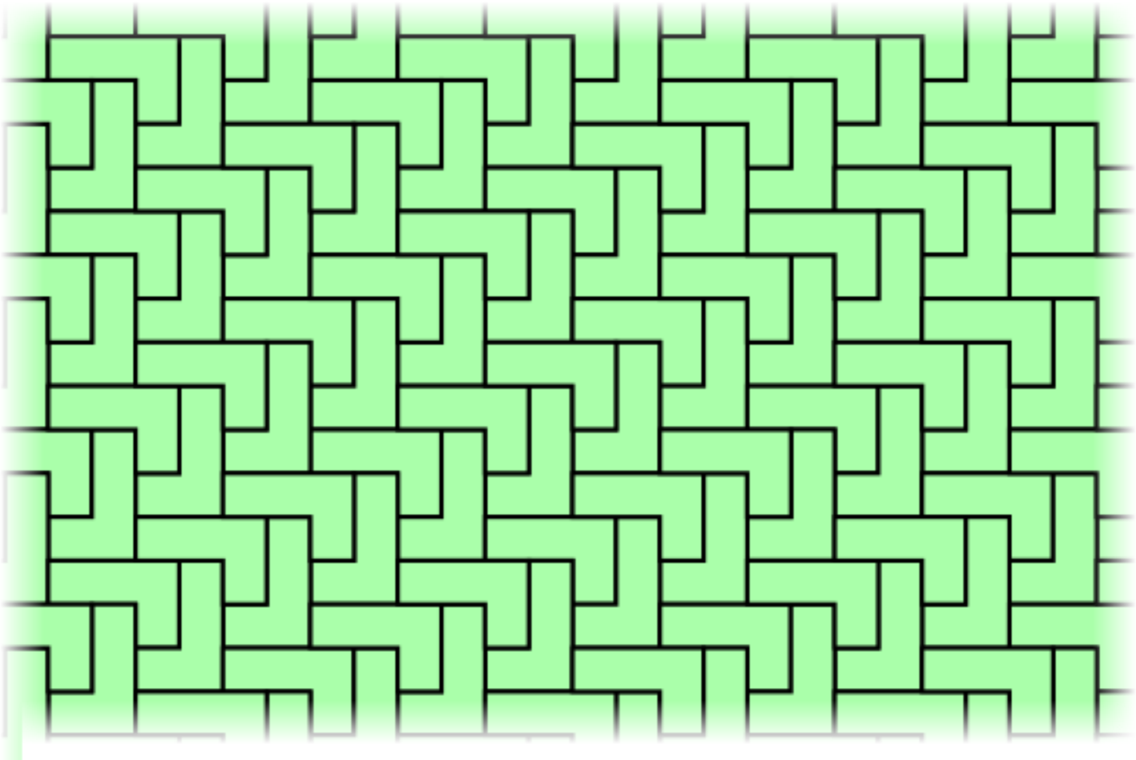
A *tiling* of $\mathbb{R}^d$ by $\{T_1, \ldots, T_k\}$ is a collection $\{S_n\}_{n=1}^{\infty}$ of tiles such that:

1. $\mathbb{R}^d = S_1 \cup S_2 \cup \cdots$,
2. $S_i^\circ \cap S_j^\circ = \emptyset$ for all $i \neq j$, and
3. For each n, there exists i and a symmetry σ of $\mathbb{R}^d$ such that $S_n = \sigma(T_i)$.

E.g.

Tile:

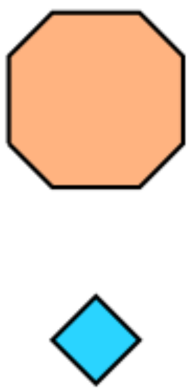
Tiling:



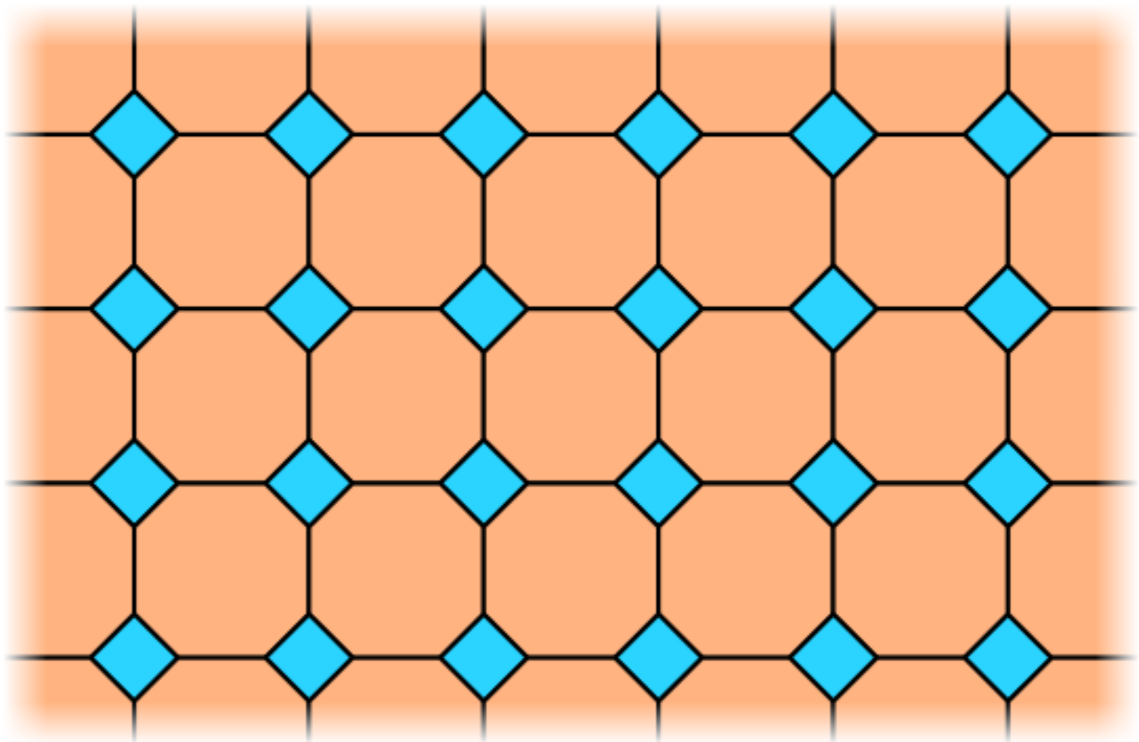
Tiles and tilings

E.g.

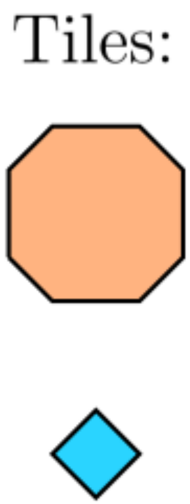
Tiles:

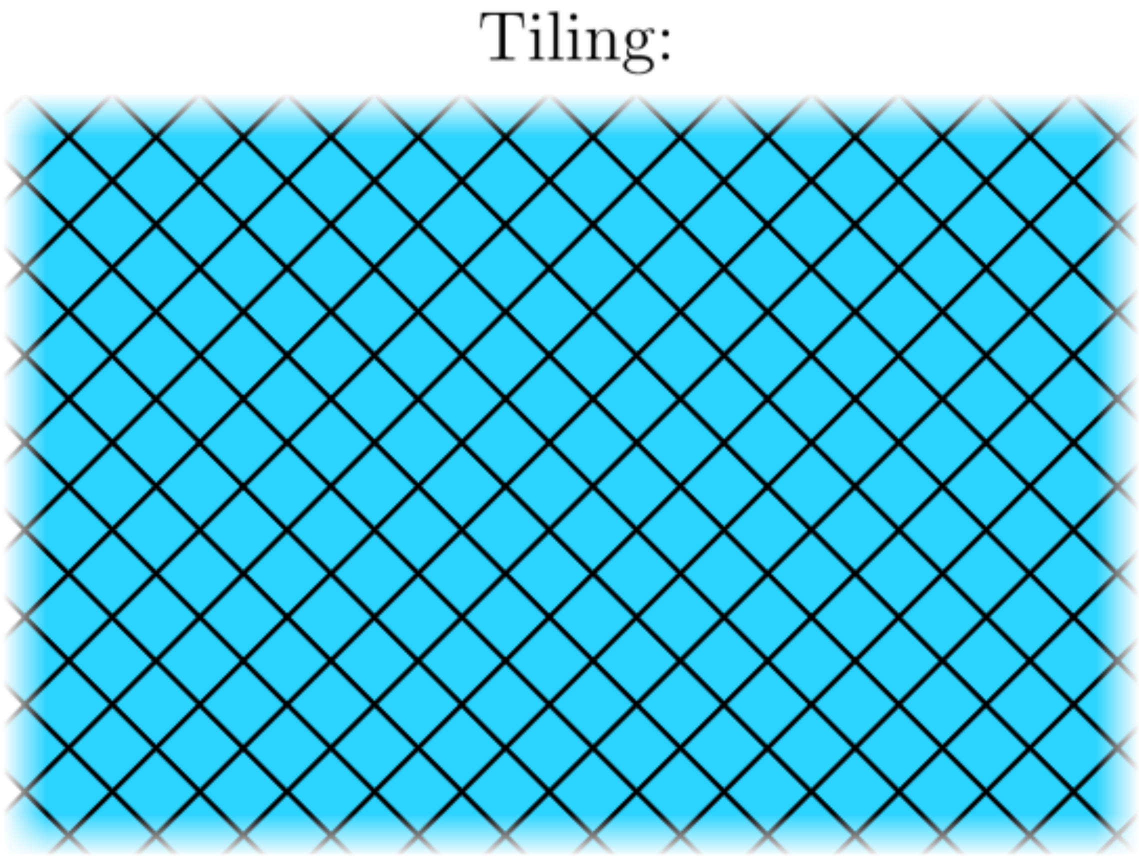


Tiling:



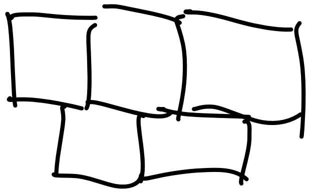
E.g.





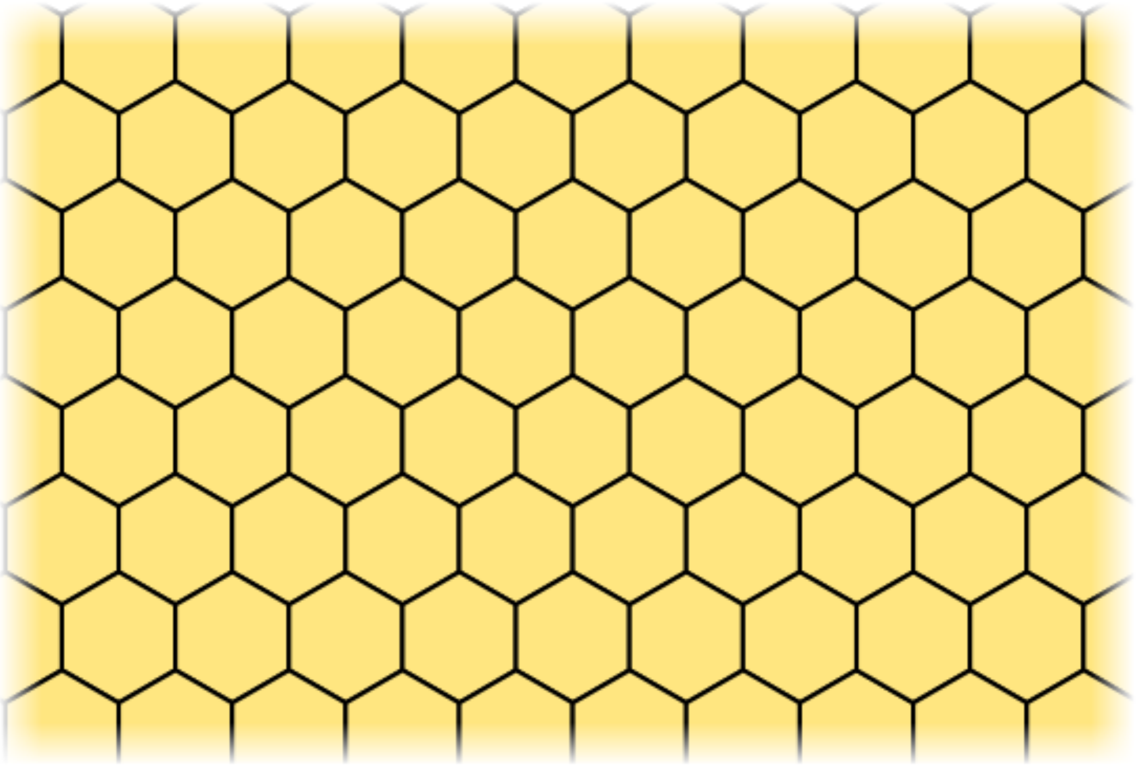
Tiles and tilings

E.g.



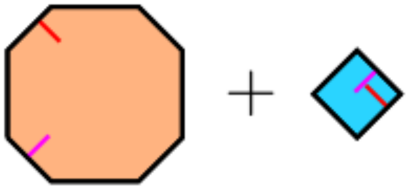
Tiling:

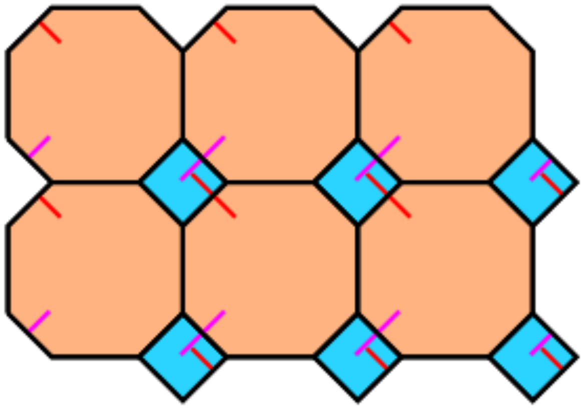
Tile:



Tiles and tilings

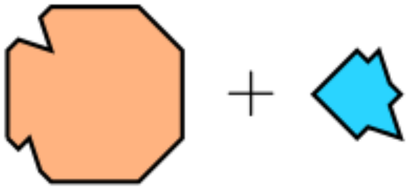
Sometimes tiles are marked to indicate matching rules.

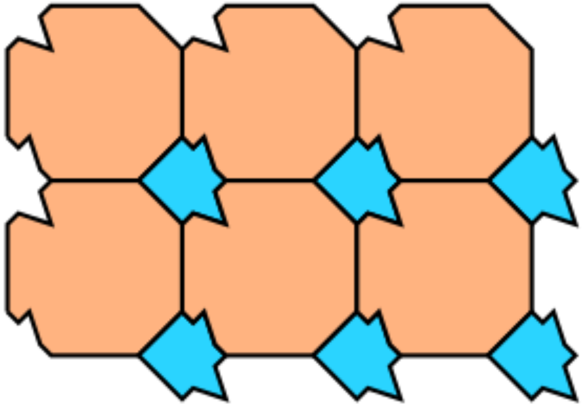




Tiles and tilings

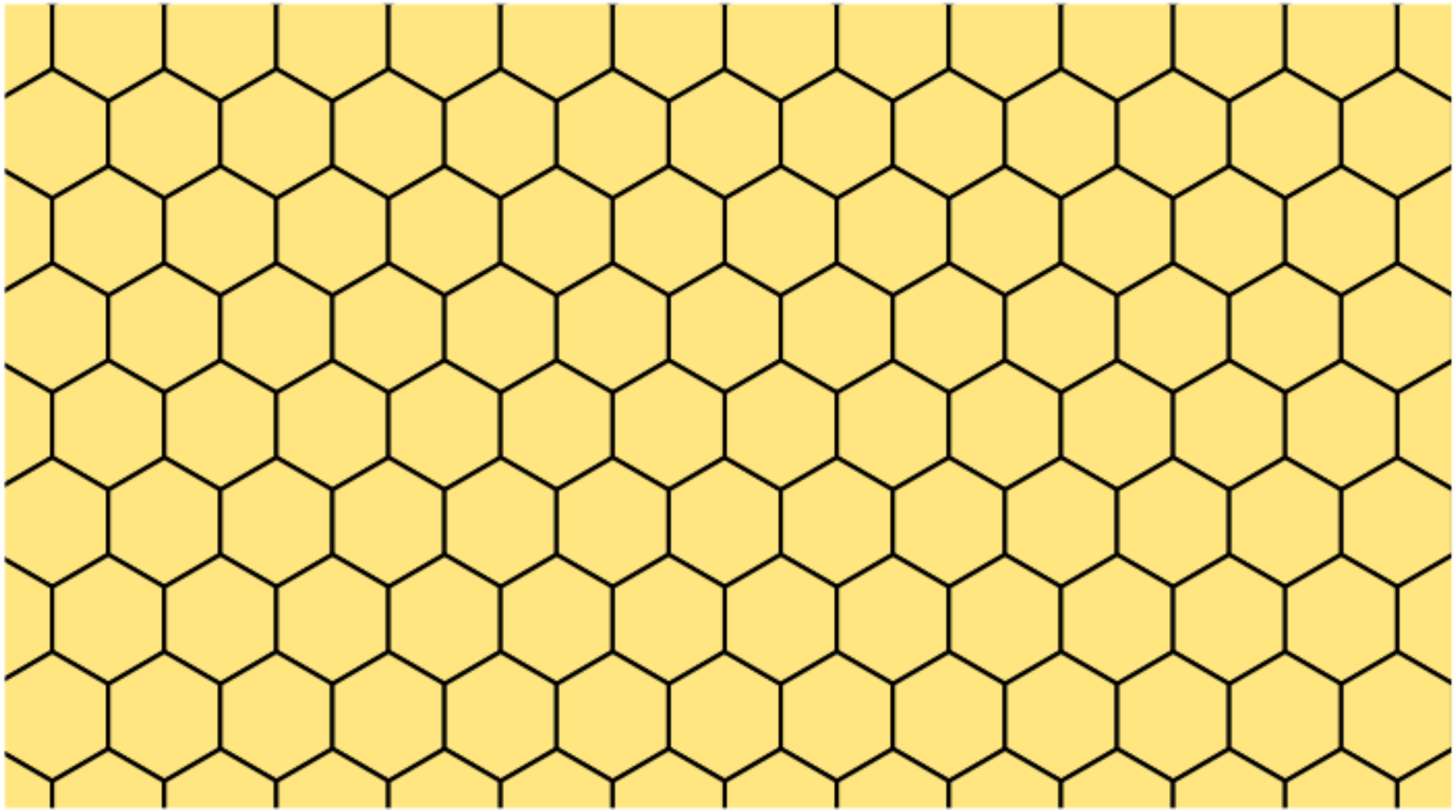
However, markings can be replaced with notches.





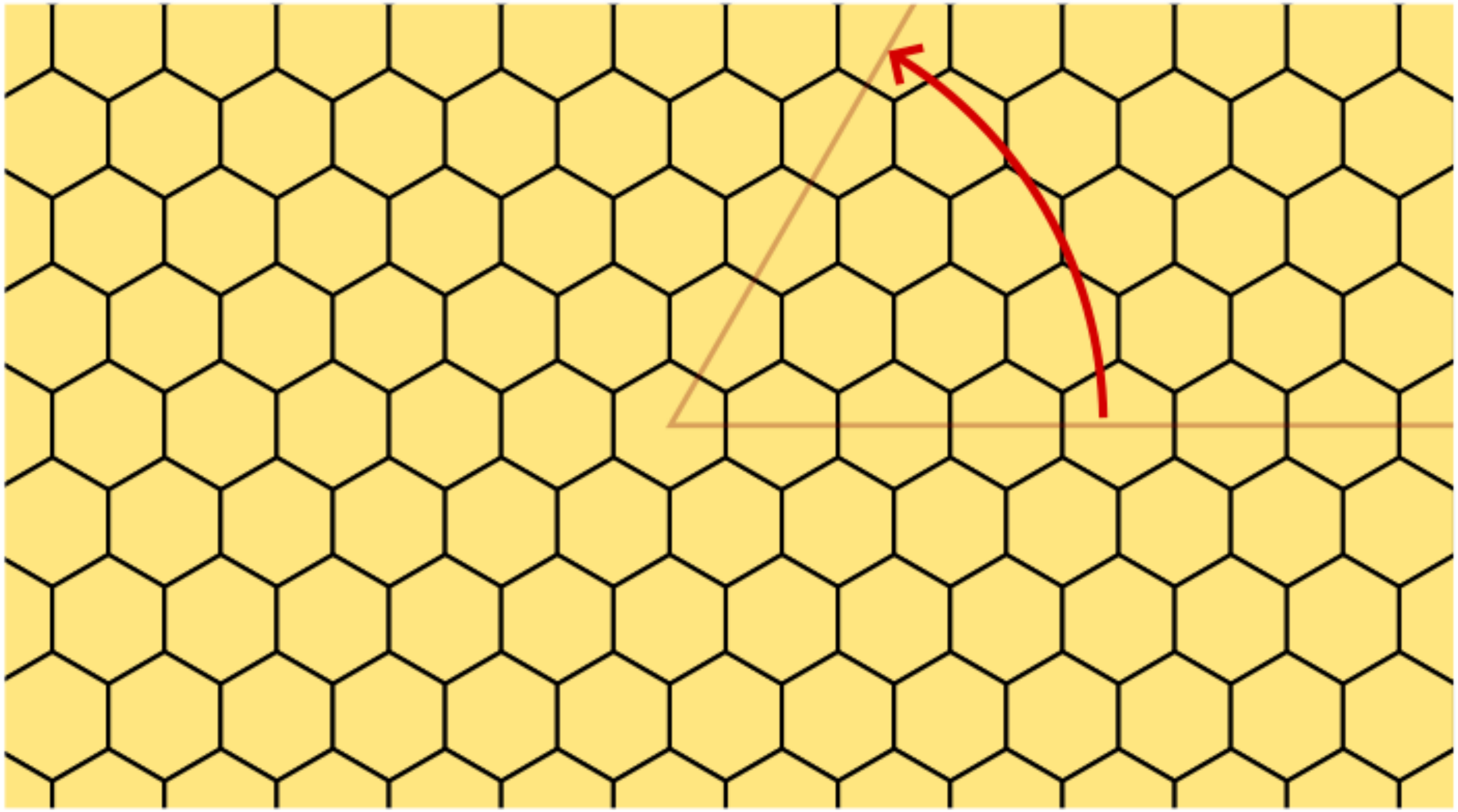
Symmetry

Tilings often have symmetry: rotations, translations, reflections which preserve the tiling.

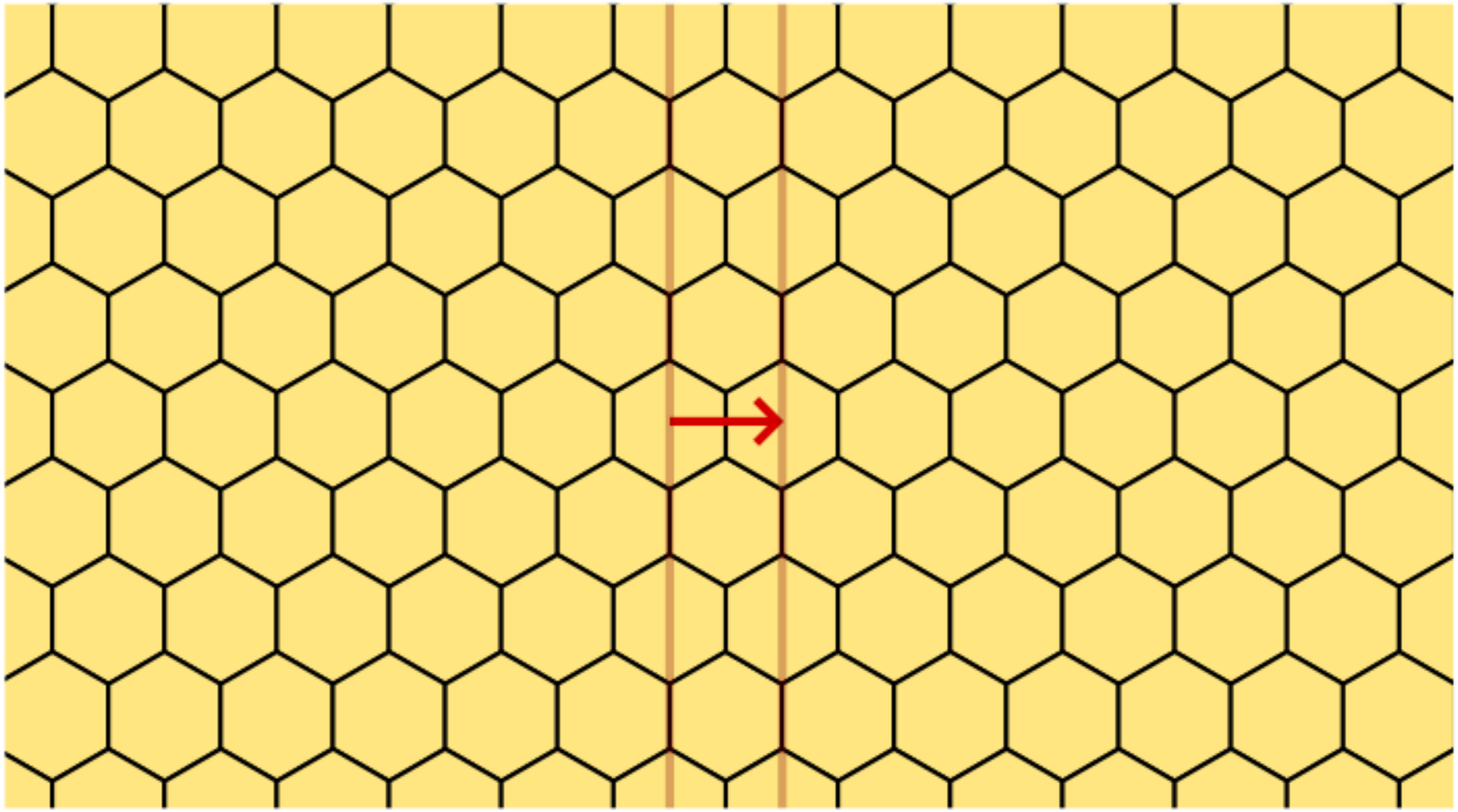


Symmetry

E.g. 6-fold rotation symmetry



E.g. translation symmetry

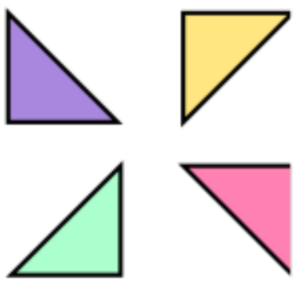


Theorem

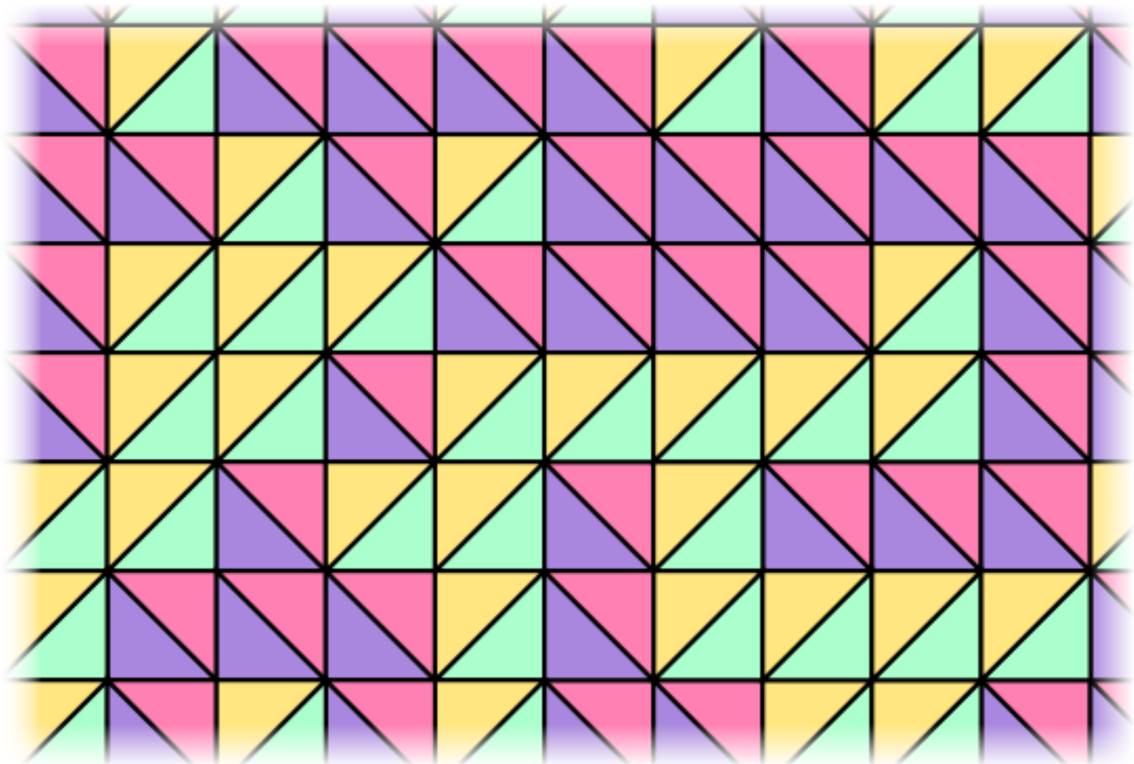
If a tiling of $\mathbb{R}^2$ has translational symmetry and rotation symmetry, then the rotation symmetry must be 2-fold (i.e., 180°), 3-fold (i.e., 120°), 4-fold (i.e., 90°), or 6-fold (i.e., ~~3~~60°).

A tiling is *aperiodic* if it has no translational symmetry.
E.g.

Tiles:



Tiling:



The tiling problem

Given a set of tiles $\{T_1, \ldots, T_k\}$, we can ask some questions:

Questions

- Does there exist a tiling using $\{T_1, \ldots, T_k\}$? (The *tiling problem*.)
- Is there a periodic tiling?

The tiling problem

Wang's trichotomy (Wang '61)

Given a set $\{T_1, \ldots, T_k\}$ of tiles, there are three possibilities:

- There is no tiling.
- There is a periodic tiling. (translational symmetry)
- There is a tiling, but all tilings are aperiodic. In this case the tile set is called *aperiodic*

Can the third option occur? If not, then the tiling problem is *decidable*: there is an algorithm to solve the tiling problem.

The tiling problem

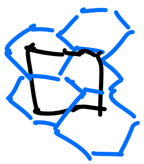
Theorem

If there is no aperiodic tile set, then the tiling problem is decidable.

Does This Tile The Plane? $(\{T_1, \ldots, T_k\})$:

  for $n = 1$ to ∞:

    Consider every patch of tiles that (just) covers $[0,n] \times [0,n]$



    If there are no such patches:

      Return NO

    If one of these patches is periodic:

      Return YES

The tiling problem

Theorem (Berger '64)

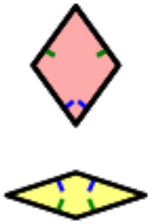
The tiling problem is not decidable. Therefore there exists an *aperiodic* set of tiles: $\{T_1, \ldots, T_k\}$ which tile the plane, but only aperiodically.

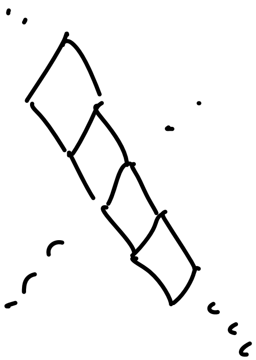
From his proof, an actual set $\{T_1, \ldots, T_k\}$ can be extracted – but with $k = 20,426$!

(! = emphasis, not factorial)

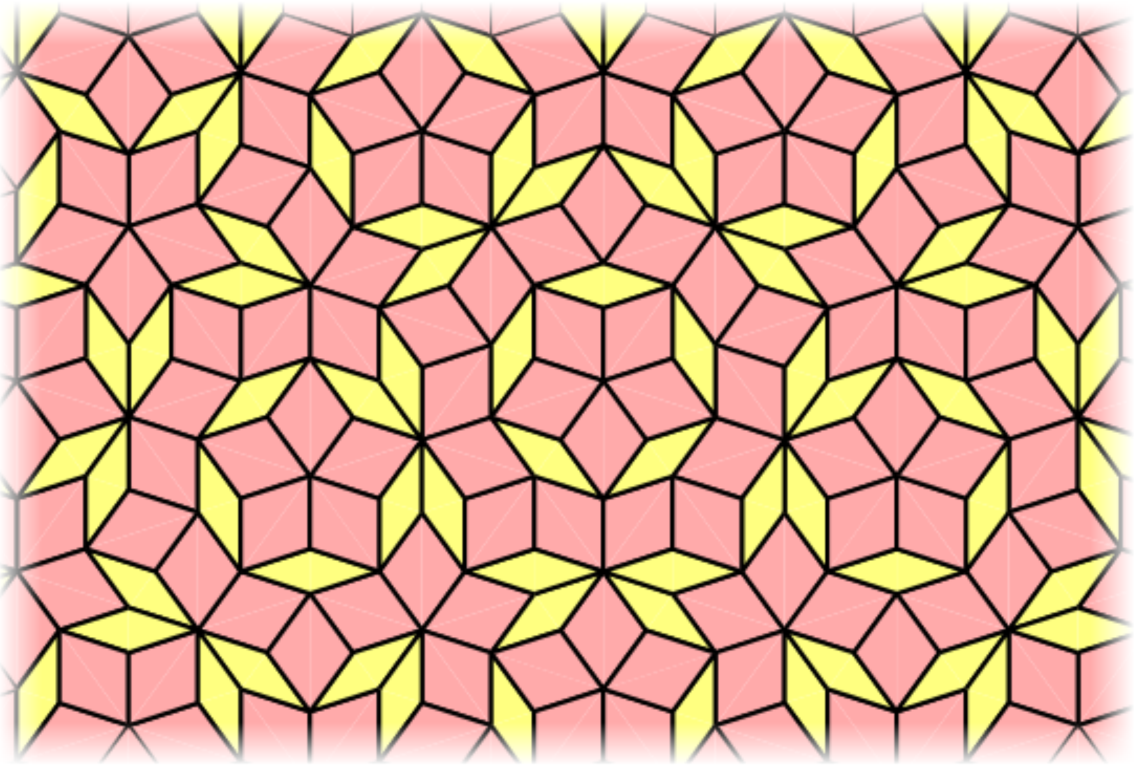
Robinson ′71: 6 tiles. Penrose ′74, Amman ′76: 2 tiles.

Tiles:





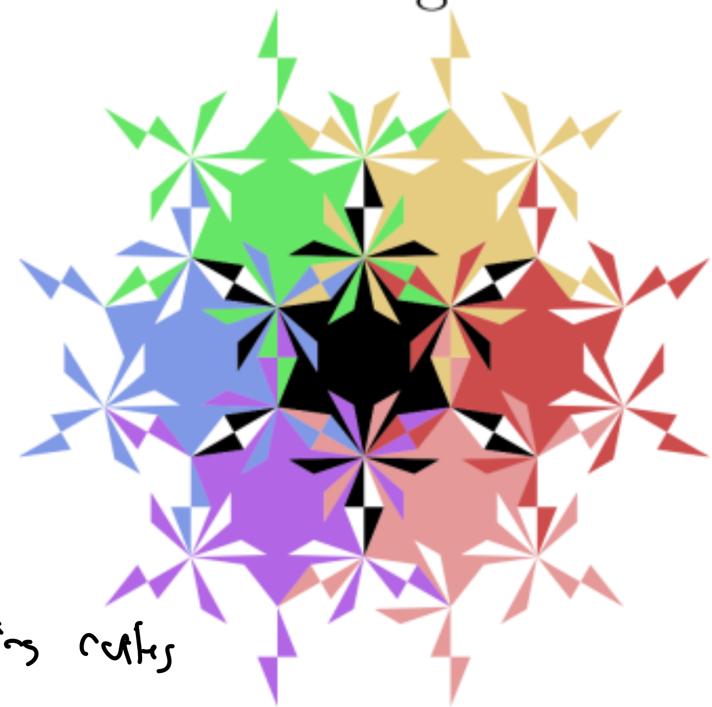
Tiling:



Socolar, Tayler ’12: 1 tile (including reflections). But...

Tile:

Tiling:



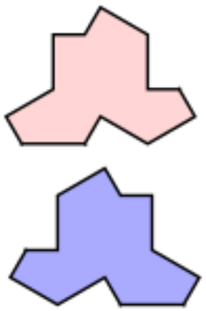
hexagons with matching rules

Images from: J.E.S. Socolar and J.M. Taylor, *Forcing nonperiodicity with a single tile*, Math Intel. 34 (2012), 18–28.

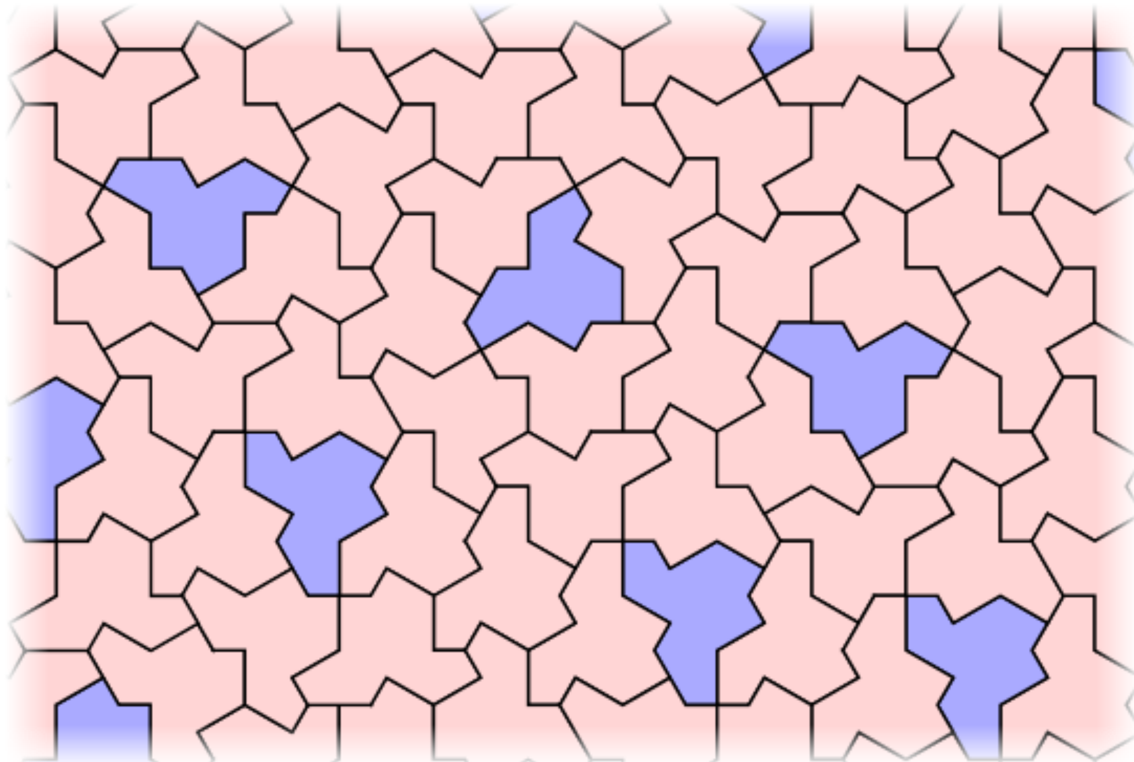
Small aperiodic sets

Smith, Myers, Kaplan, Goodman–Strauss '23: 1 tile

Tiles:



Tiling:



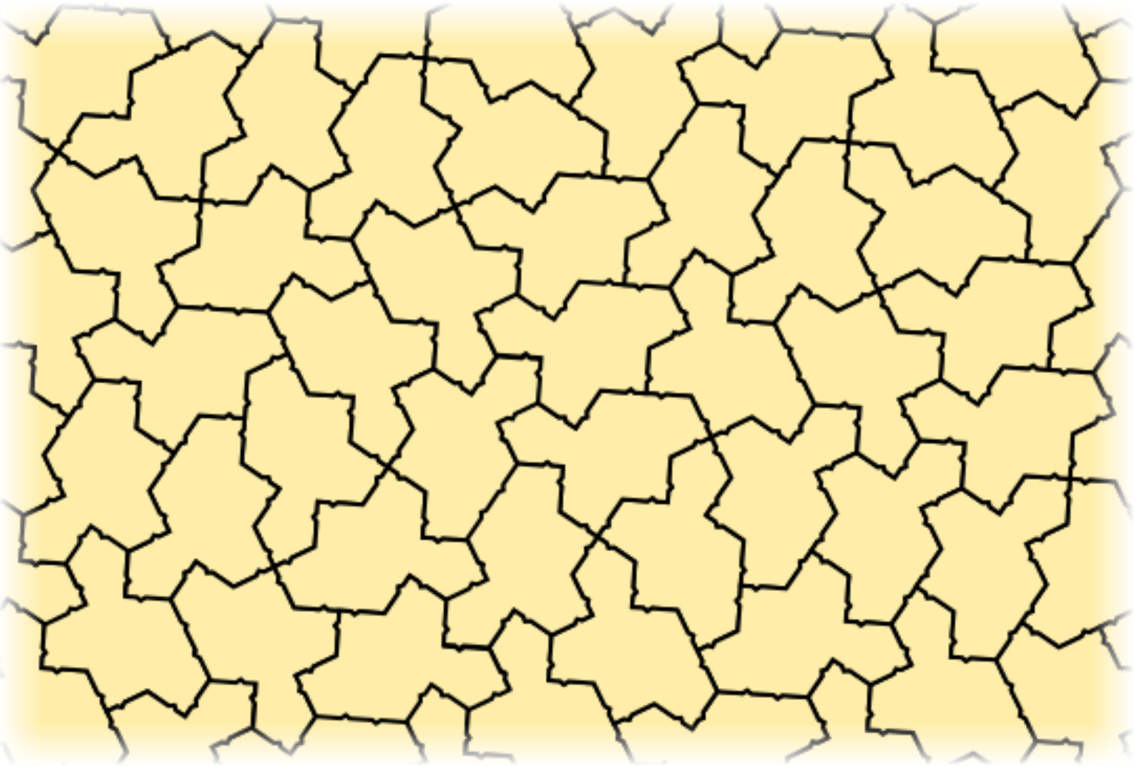
"the hat"

(reflection needed)

Smith, Myers, Kaplan, Goodman–Strauss ’23: 1 tile

Tile:

Tiling:



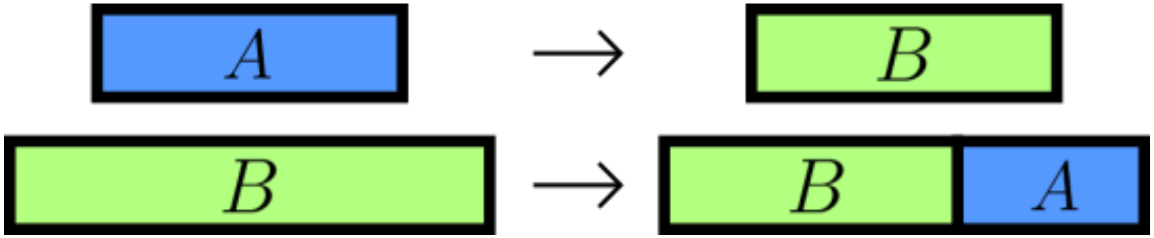
(no reflection needed – or even possible!)

How do we know that these tiles can cover the plane? And that there is no periodic covering?

In most cases, tilings are proven to have a “heirarchical” structure, which can be captured through a construction called **substitution tilings**.

Substitution tilings

E.g. 1-dimensional Fibonacci tiling



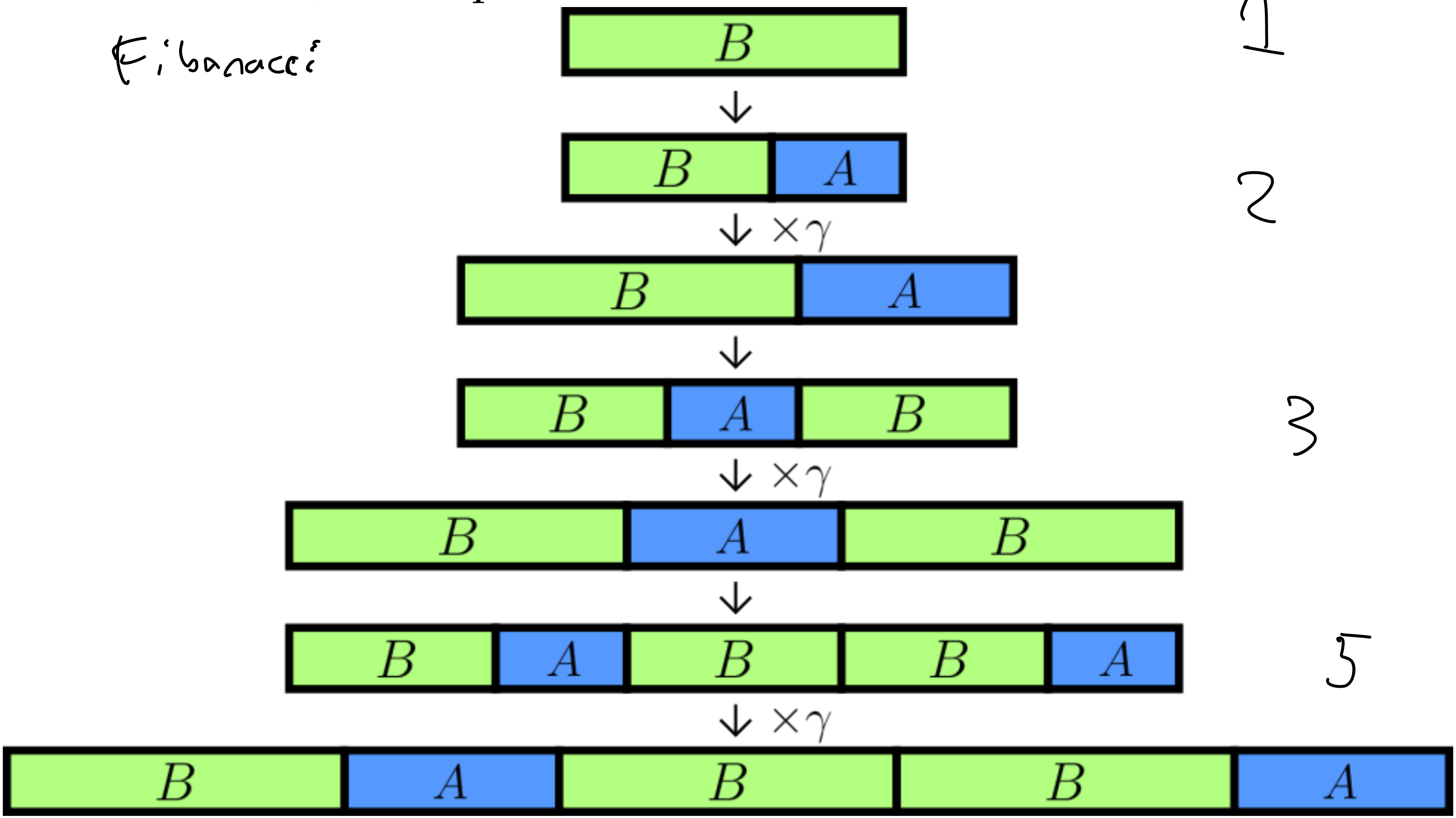
Here we have two different tiles, along with a rule that tells us to replace each tile with a shrunken patch of other tiles.

Shrinking is by a fixed factor: $\gamma = \frac{1+\sqrt{5}}{2}$.

Substitution tilings

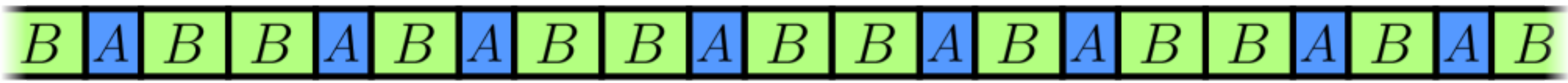
To generate the tiling, we apply the substitution rule, stretch by the factor γ, and repeat.

Fibonacci



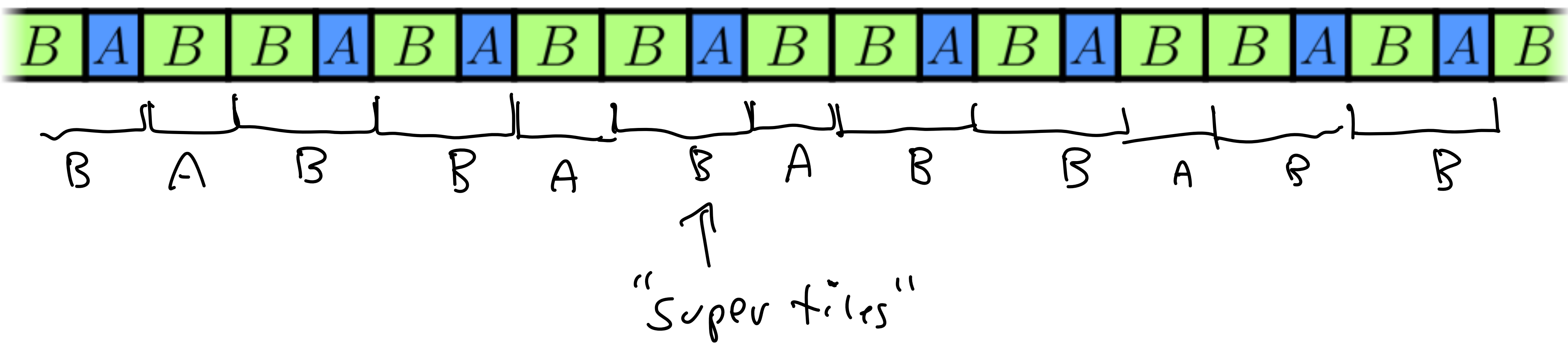
Substitution tilings

If we line up each stage appropriately, then the tiling grows and covers the whole line in the limit.



Substitution tilings

Given a tiling using the Fibonacci substitution, we can undo the substitution.

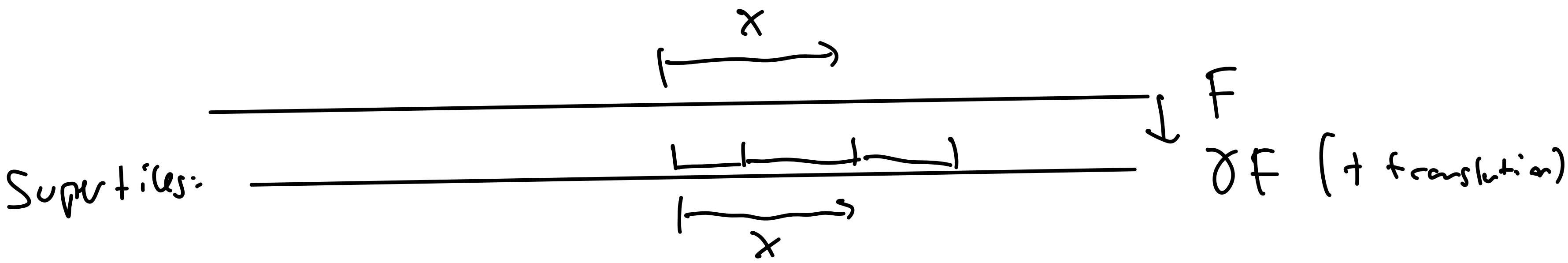


Substitution tilings

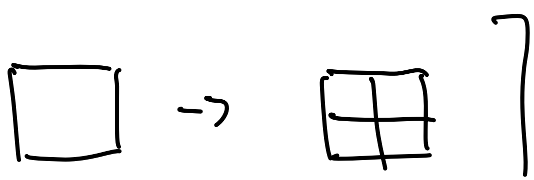
Theorem

The Fibonacci tiling is aperiodic.

Pf: By contradiction. Suppose $x > 0$ is the minimal translational symmetry.



$\therefore$ F has translational symmetry by $\frac{1}{\gamma} x < x$



Contradiction! $\square$

Substitution tilings

The same idea can be used more generally:

Theorem

Given a tiling substitution rule for a set of tiles $\{T_1, \ldots, T_k\}$ in $\mathbb{R}^d$. If any tiling of $\mathbb{R}^d$ comes from a unique substitution (i.e., there is one and only one way to identify supertiles), then the tile set is aperiodic.

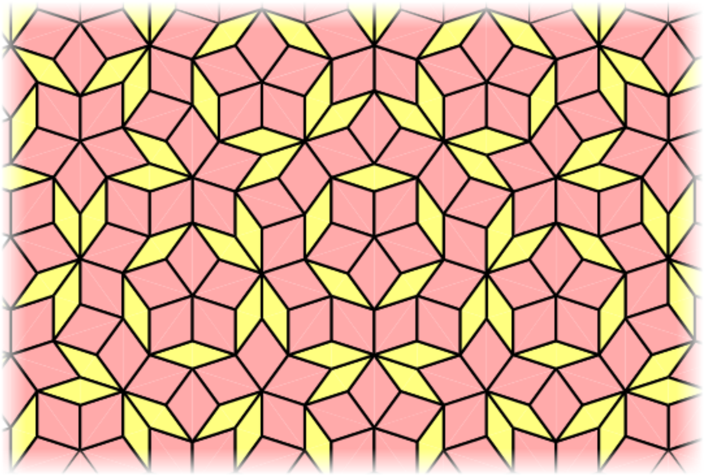
Substitution tilings

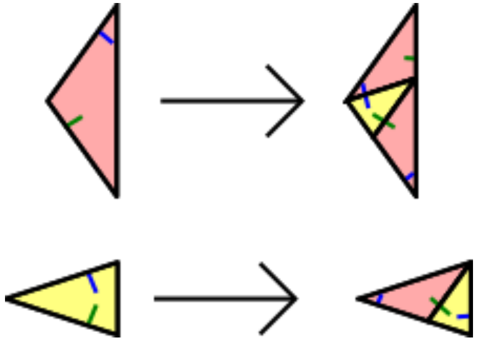
Evidence suggests the idea of using substitutions to create aperiodic tilings was found by Islamic mathematicians/ architects over 500 years ago, and used in (some) girih decoration.

From the Darb-e Imam shrine (Isfahan, Iran):

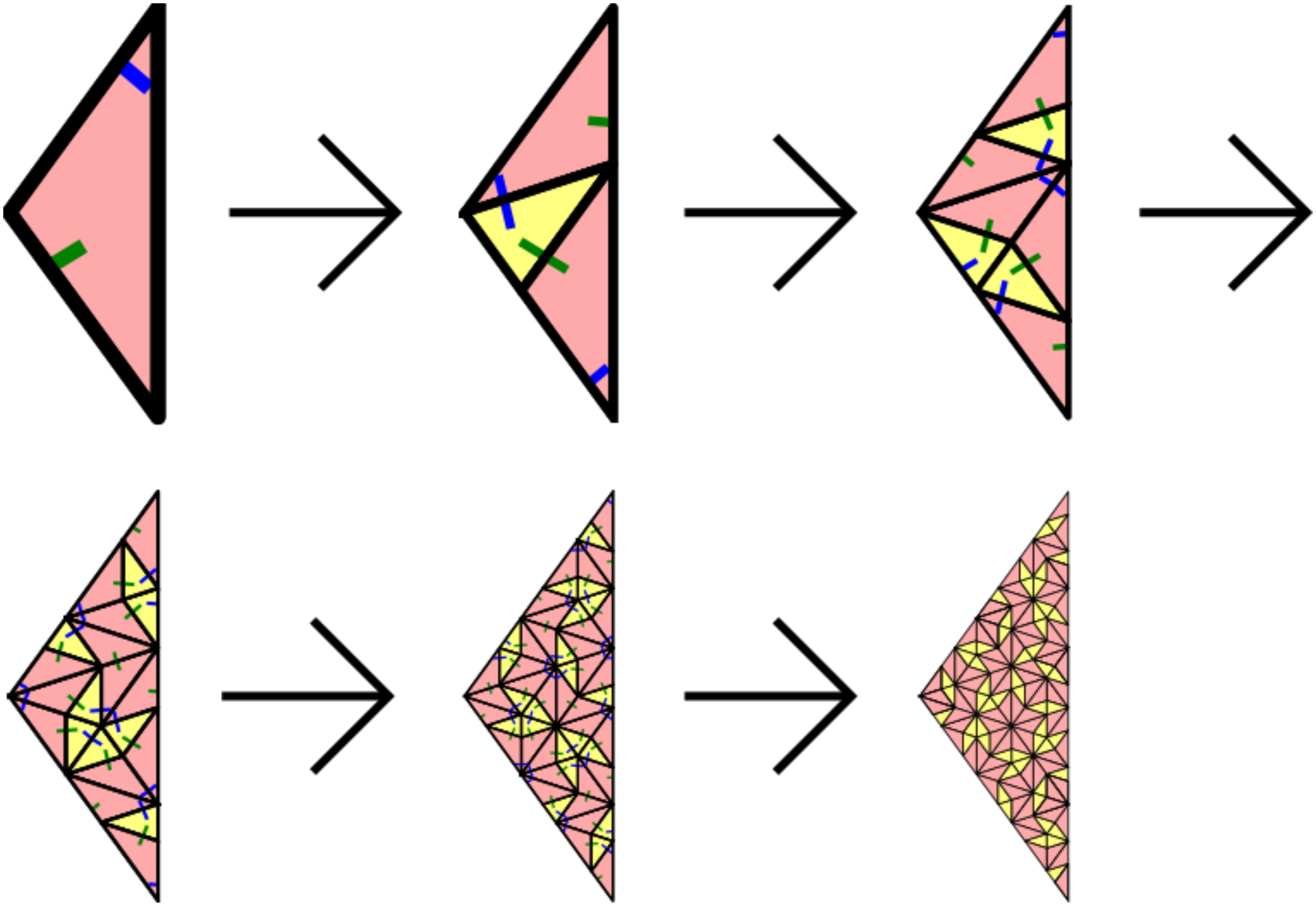
Substitution tilings

Often we need to combine/decompose the original tiles, before we can find a substitution rule.

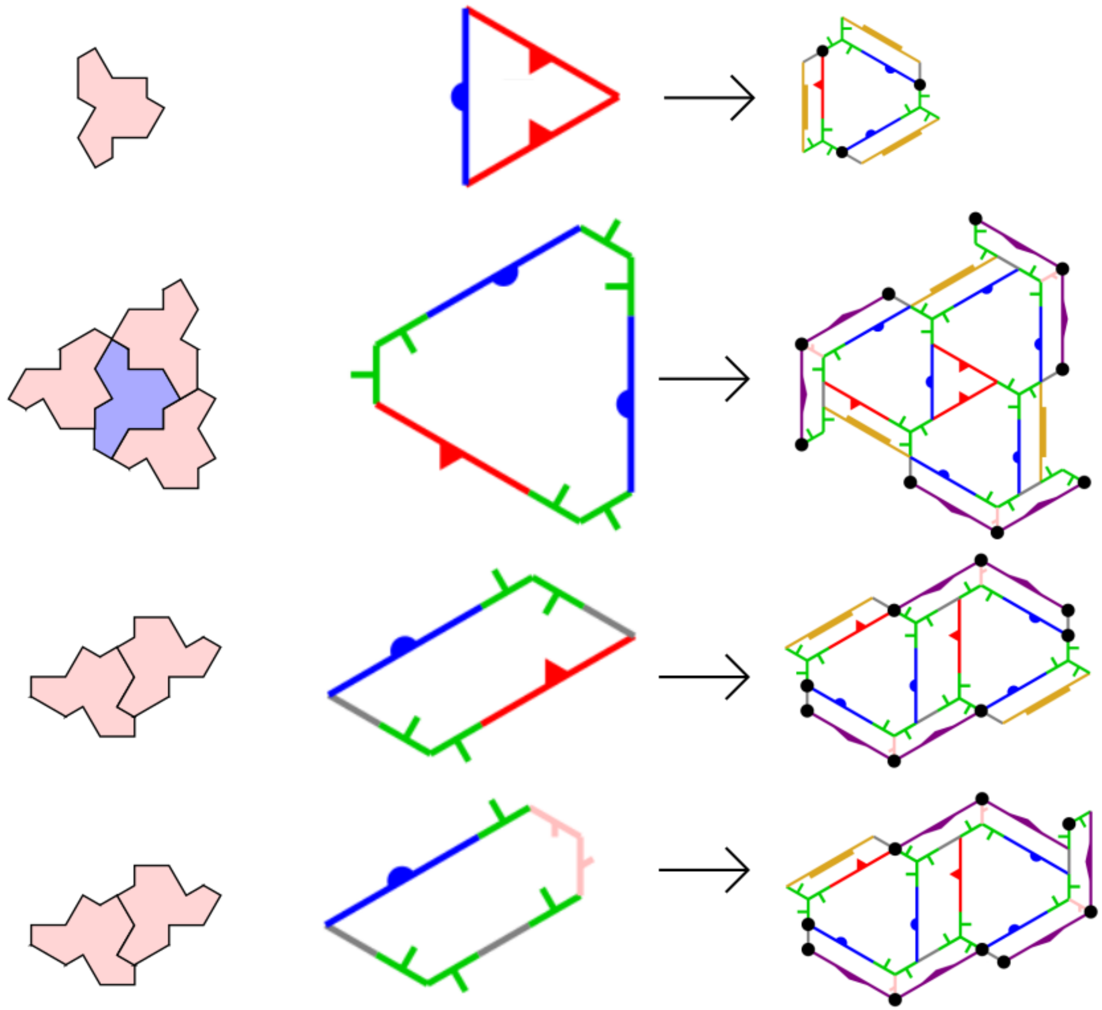




Substitution tilings



Substitution tilings



Images from: D. Smith, J.S. Myers, C.S. Kaplan, C. Goodman-Strauss, *An aperiodic monotile*, arXiv:2303.10798.

Quasicrystals

Do aperiodic tilings occur in the real world?

In the ’80s, unusual symmetries were observed in diffraction patterns of certain rapidly-cooled alloys:

- Schechtman ’82: 10-fold symmetry in Al-Mn. (2011 Nobel prize.)
- Ishimasa, Nissen, Fukano ’85: 12-fold symmetry in Ni-Cr.
- Wang, Chen, Kuo ’87: 8-fold symmetry in V-Ni-Si, Cr-Ni-Si.

These are molecular structures with order, but not translational symmetry. In some cases, specific aperiodic tilings (such as the Penrose–Amman tiling) are believed to model the structure at the molecular level.

Quasicrystals

In the *real* real world: various quasicrystals (including "icosohedrite", with 10-fold symmetry) have been found in a meteorite (Bindi, Steinhardt, et al., 2011).

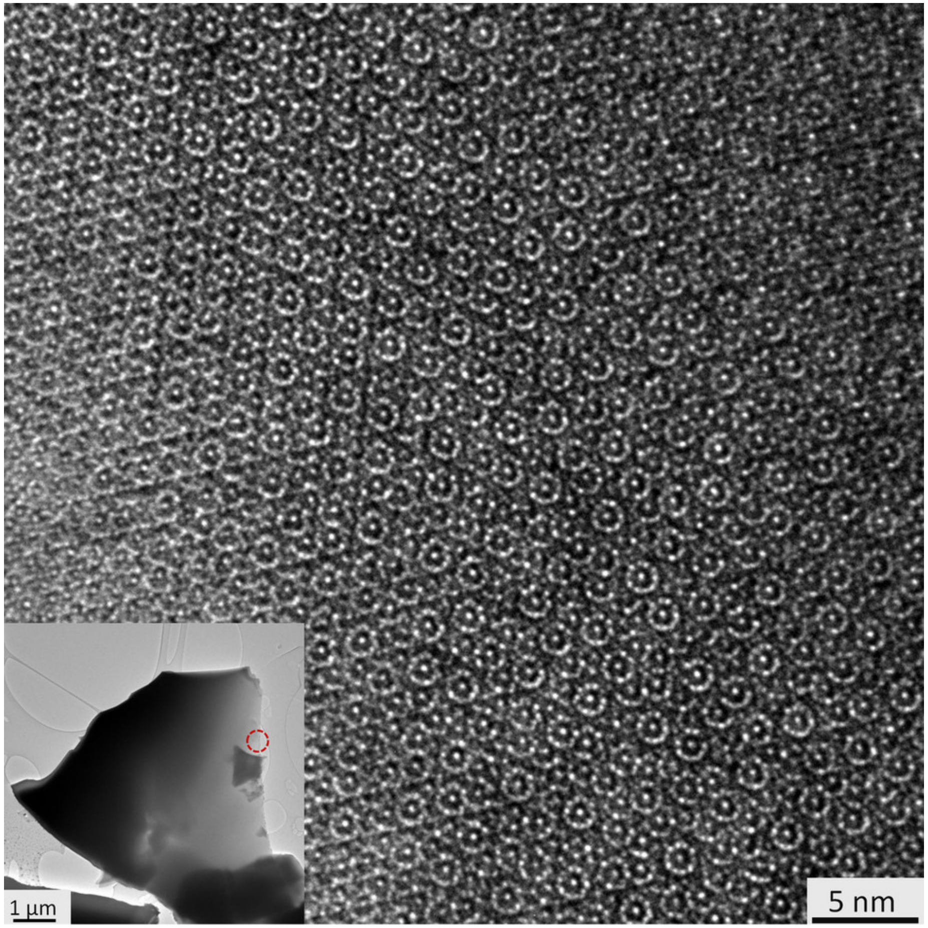


Image from: Bindi et al., *Natural quasicrystal with decagonal symmetry*, Sci. Rep. 5, (2015), article 9111.