KU LEUVEN

Nonsingular Bernoulli actions: a survey

Canadian Operator Algebras Symposium

Ottawa, 30 May – 3 June 2022

Stefaan Vaes – KU Leuven

Nonsingular Bernoulli shifts

Main question

Which injective factors M are of the form $M \cong L^\infty(X,\mu) \rtimes_T \mathbb{Z}$ where

$$\mathbb{Z} \curvearrowright^T (X,\mu) = \prod_{n\in\mathbb{Z}} (X_0, \mu_n) : (Tx)_n = x_{n-1}$$

is a nonsingular Bernoulli shift ?

- Fixed base space X_0, with variable probability measures μ_n.
- All $\mu_n \sim \mu_0$ are equivalent, and we need a condition to ensure nonsingularity.
- By Connes and Haagerup classification, the question is: what are the possible types and associated flows of Bernoulli shifts ?
- Obvious: taking $\mu_n = \mu_0$ for all n, we get the hyperfinite II_1 factor.

KU LEUVEN

Nonsingular Bernoulli shifts

Main question

Which injective factors M are of the form $M \cong L^\infty(X,\mu) \rtimes_T \mathbb{Z}$ where

$$\mathbb{Z} \curvearrowright^T (X,\mu) = \prod_{n\in\mathbb{Z}} (X_0,\mu_n) : (Tx)_n = x_{n-1}$$

is a nonsingular Bernoulli shift ?

- Hamachi, 1981: type III may arise.
- Kosloff, 2009: type III_1.
- Björklund-Kosloff-V, 2019: never of type II_∞ if $X_0 = \{0,1\}$.
- Kosloff-Soo, 2020 and Berendschot-V, 2020: type III_λ for all $\lambda \in (0,1]$.
- Berendschot-V, 2020: type II_∞ and some type III_0.
- Berendschot-V, 2021: **not all type III_0 can be attained! But many can.**

KU LEUVEN

Krieger type and associated flow

Consider an essentially free, ergodic, nonsingular action $G \curvearrowright (X, \mu)$, with μ nonatomic.

- **Type II_1 :** there exists a G-invariant probability measure $\nu \sim \mu$.
- **Type II_∞ :** there exists a G-invariant infinite measure $\nu \sim \mu$.
- **Type III :** the others.

Denote $\omega(g, x) = \log \frac{d(g^{-1} \cdot \mu)}{d\mu}(x)$. Then $\omega : G \times X \to \mathbb{R}$ is a 1-cocycle.

Maharam extension: $G \curvearrowright X \times \mathbb{R} : g \cdot (x, s) = (g \cdot x, \omega(g, x) + s)$, which commutes with $\mathbb{R} \curvearrowright X \times \mathbb{R} : t \cdot (x, s) = (x, t + s)$.

$\rightsquigarrow$ Ergodic decomposition $\pi : X \times \mathbb{R} \to Y$ of the Maharam extension, with $\mathbb{R} \curvearrowright Y$.

Krieger's associated flow; also **Connes-Takesaki flow of weights** of $L^\infty(X) \rtimes G$.

Krieger type and associated flow

Consider an essentially free, ergodic, nonsingular action $G \curvearrowright (X, \mu)$, with μ nonatomic.

Let $\mathbb{R} \curvearrowright (Y, \eta)$ be the associated flow, which is ergodic.

- Type II_1 or type II_∞ iff the associated flow is $\mathbb{R} \curvearrowright \mathbb{R}$ by translation.
- Type III_1 iff Y is one point.
- Type III_λ with $0 < \lambda < 1$ iff the associated flow is periodic: $\mathbb{R} \curvearrowright \mathbb{R}/\mathbb{Z}\log\lambda$.
- Type III_0 iff the associated flow is properly ergodic.

$\rightsquigarrow$ The types and the associated flow are invariants of the crossed product factors.

$\rightsquigarrow$ Krieger, Connes, Takesaki, Haagerup: complete invariants if G is amenable.

Back to Bernoulli shifts

- Let $\mathbb{Z} \curvearrowright^T (X,\mu) = \prod_{n\in\mathbb{Z}}(X_0,\mu_n) : (Tx)_n = x_{n-1}$ be a nonsingular Bernoulli shift.
- Assume that μ is nonatomic and that $\mathbb{Z} \curvearrowright^T (X,\mu)$ is ergodic (which is not automatic).

Results from Berendschot-V, 2021 :

Obstruction

The associated flow $\mathbb{R} \curvearrowright (Y,\eta)$ must have a property of **infinite divisibility**. Not all ergodic flows have this property.

Construction

Introduction of **Poisson flows**, which include all **almost periodic flows**, and proof that every Poisson flow arises as associated flow of a nonsingular Bernoulli shift.

Also: connection with ITPFI_2 factors.

Joint flows and infinite divisibility

Construction of Hamachi, Oka, Osikawa, 1974:

- Let $\mathbb{R} \curvearrowright (Y_i, \eta_i)$ be ergodic flows, with $i = 1, \ldots, n$.
- Consider the product action of $H = \mathbb{R}^n$ on $(Y, \eta) = (Y_1 \times \cdots \times Y_n, \eta_1 \times \cdots \times \eta_n)$.
- Define the subgroup $H_0 < H$ as the kernel of $H \to \mathbb{R} : (t_1, \ldots, t_n) = t_1 + \cdots + t_n$.

$\rightsquigarrow$ The **joint flow** is the ergodic flow $\mathbb{R} = H/H_0 \curvearrowright L^\infty(Y, \eta)^{H_0}$.

Observation

The flow of weights of a tensor product $M_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} M_n$ is the joint flow of the flows of weights of the M_i.

Definition: we say that an ergodic flow $\mathbb{R} \curvearrowright (Y, \eta)$ is **infinitely divisible** if it is, for every $n \in \mathbb{N}$, isomorphic with the joint flow of n identical ergodic flows.

Bernoulli shifts: nonsingular, nonatomic, not dissipative

Kakutani: the shift $\mathbb{Z} \curvearrowright^{T} (X, \mu) = \prod_{n \in \mathbb{Z}} (X_0, \mu_n)$ is nonsingular iff

- $\mu_n \sim \mu_0$ for all $n \in \mathbb{Z}$, and
- $\sum_{n \in \mathbb{Z}} H^2(\mu_{n+1}, \mu_n) < +\infty$.

Here: $H^2(\mu, \mu') = \frac{1}{2} \int_{X_0} \left| \sqrt{d\mu/d\zeta} - \sqrt{d\mu'/d\zeta} \right|^2 d\zeta$ is the Hellinger distance.

For a nonsingular Bernoulli shift, **trivialities** may occur:

- μ could be **atomic**.
- The action could be **dissipative**, i.e. admit a fundamental domain, i.e. be of type I.

  Example: $\mu_n = \mu_-$ for all $n < 0$ and $\mu_n = \mu_+$ for all $n \geq 0$, and $\mu_- \neq \mu_+$.

$\rightsquigarrow$ We will assume that we are not in one of these trivial situations.

KU LEUVEN

Obstruction: infinite divisibility

Let $\mathbb{Z} \curvearrowright^T (X,\mu) = \prod_{n\in\mathbb{Z}}(X_0,\mu_n)$ be an arbitrary nonsingular Bernoulli shift.

Assume that μ is not atomic and that the action does not admit a fundamental domain.

Theorem (Berendschot-V, 2021)

There exists a unique $C \subset X_0$ with $\sum_n(1-\mu_n(C)) < +\infty$ such that

- the action on $C^{\mathbb{Z}} \subset X$ is ergodic and its associated flow is infinitely divisible,
- the action on $X \setminus C^{\mathbb{Z}}$ admits a fundamental domain.

Conclusion: for every ergodic nonsingular Bernoulli shift, the associated flow is infinitely divisible.

Giordano-Skandalis-Woods, 1984: not every ergodic flow is infinitely divisible.
Even more: there are ITPFI factors whose flow of weights cannot be divided by two.

Main idea to prove infinite divisibility

Let $\mathbb{Z} \curvearrowright^T (X,\mu) = \prod_{n\in\mathbb{Z}}(X_0,\mu_n)$ be an arbitrary nonsingular Bernoulli shift.

Assume that μ is not atomic and that the action is conservative (i.e. recurrent).

- Denote by $\mathcal{S}$ the group of finite permutations of the set $\mathbb{Z}$. Note that we have the nonsingular action $\mathcal{S} \curvearrowright (X,\mu)$.
- Consider the Maharam extensions $\mathbb{Z} \curvearrowright X \times \mathbb{R}$ and $\mathcal{S} \curvearrowright X \times \mathbb{R}$
- By careful approximation, with ideas going back to Kosloff and Danilenko : $L^\infty(X\times\mathbb{R})^{\mathbb{Z}} = L^\infty(X\times\mathbb{R})^{\mathcal{S}} = L^\infty(X\times\mathbb{R})^{2\mathbb{Z}}$.

Write $E = 2\mathbb{Z}$ and $O = 1+2\mathbb{Z}$, with finite permutations $\mathcal{S}_E$ and $\mathcal{S}_O$.

$\rightsquigarrow$ $L^\infty(X\times\mathbb{R})^{\mathbb{Z}} = L^\infty(X\times\mathbb{R})^{\mathcal{S}} \subset L^\infty(X\times\mathbb{R})^{\mathcal{S}_E\times\mathcal{S}_O} = L^\infty(X\times\mathbb{R})^{2\mathbb{Z}\times 2\mathbb{Z}}$
$\subset L^\infty(X\times\mathbb{R})^{2\mathbb{Z}} = L^\infty(X\times\mathbb{R})^{\mathbb{Z}}$.

So all are equal and we have divided the flow by two.

Intermezzo: tail boundary flows

Let $(\eta_n)_{n\geq 1}$ be probability measures on $\mathbb{R}$.

- Consider a random walk on $\mathbb{R}$ with independent steps having distribution η_n.
- Note: if $\eta_0 \sim$ Lebesgue is the initial distribution, then the distribution at step n is $\eta_0 * \eta_1 * \cdots * \eta_n$.
- **Connes-Woods, 1988** : Poisson boundary $\mathbb{R} \curvearrowright (Y, \eta)$ of this random walk.

  These are **precisely** the flows of weights of ITPFI factors $\overline{\bigotimes}_{n\in\mathbb{Z}}(B(H_n), \varphi_n)$.

  These are **precisely** the approximately transitive flows.

$\leadsto$ We call this Poisson boundary $\mathbb{R} \curvearrowright (Y, \eta)$ the **tail boundary flow** of $(\eta_n)_{n\geq 1}$.

Construction: Bernoulli shifts with Poisson associated flow

Let $\mathbb{Z} \curvearrowright^T (X, \mu) = \prod_{n \in \mathbb{Z}} (X_0, \mu_n)$ be a nonsingular Bernoulli shift, with μ not atomic, and T conservative (i.e. recurrent). We have already seen that T is then ergodic.

Lemma (Berendschot-V, 2020)

If there exists a measure $\nu \sim \mu_0$ such that for a.e. $x \in X_0$, we have that $\theta_n(x) := -\log(d\mu_n/d\nu)(x) \to 0$, then the associated flow of T is isomorphic with the tail boundary flow of the sequence $(\theta_n)_*(\mu_n)$.

- We should not naively think that all tail boundary flows can thus be attained. Indeed: some tail boundary flows are not infinitely divisible.
- Difficult tension with the requirement that T is nonsingular and conservative.
- We can realize **Poisson flows:** tail boundary flows with all η_n being Poisson distributions.

Construction: Bernoulli shifts with Poisson associated flow

We call **Poisson flow** every ergodic flow that can be realized as the tail boundary flow of (equivalently)

- a sequence of Poisson distributions η_{a_n,λ_n},
- a sequence of compound Poisson distributions $\mathcal{E}(\gamma_n)$.

Here: $\eta_{a,\lambda}$ is supported on $\{0, a, 2a, \dots\}$ with $\eta_{a,\lambda}(ka) = \exp(-\lambda)\lambda^k/k!$,

and $\mathcal{E}(\gamma) = \exp(-\|\gamma\|)\exp(\gamma)$ whenever γ is a finite positive measure on $\mathbb{R}$.

Theorem (Berendschot-V, 2021)

For every infinite amenable group G and for every Poisson flow $\mathbb{R} \curvearrowright (Y,\eta)$, there exists a nonsingular Bernoulli action $G \curvearrowright \prod_{h\in G}(X_0,\mu_h)$ that is ergodic and has associated flow $\mathbb{R} \curvearrowright (Y,\eta)$.

ITPFI$_2$ factors and Poisson flows of positive type

- Restricting to $\eta_{a,\lambda}$ with $a > 0$, or to $\mathcal{E}(\gamma)$ with γ supported on $\mathbb{R}_+$, we get: **Poisson flows of positive type.**
- Recall: ITPFI$_2$ factors are factors of the form $\overline{\bigotimes}_{n=1}^{\infty}(M_2(\mathbb{C}), \varphi_n)$.

Theorem (Berendschot-V, 2021)

The Poisson flows of positive type are precisely the flows of weights of ITPFI$_2$ factors.

Every almost periodic flow $\mathbb{R} \curvearrowright (K, \mathrm{Haar})$ is a Poisson flow of positive type.

Corollary. Let $\mathbb{R} \curvearrowright (K, \mathrm{Haar})$ be an almost periodic ergodic flow.

- For every infinite amenable G, there exists a nonsingular Bernoulli action $G \curvearrowright (X, \mu)$ with associated flow $\mathbb{R} \curvearrowright K$.
- There exists an ITPFI$_2$ factor M with flow of weights $\mathbb{R} \curvearrowright K$.

Nonsingular Bernoulli actions of nonamenable groups

Consider $G \curvearrowright (X, \mu) = \prod_{h \in G} (X_0, \mu_h)$ by $(g \cdot x)_h = x_{g^{-1}h}$.

Kakutani: the action is nonsingular iff for every $g \in G$, we have $\sum_{h \in G} H^2(\mu_{gh}, \mu_h) < +\infty$.

$\rightsquigarrow$ This gives a 1-cocycle $g \mapsto c_g \in \ell^2(G) \otimes L^2(X_0, \mu_e)$.

Proposition (V-Wahl, 2017)

If G is nonamenable and $b_1^{(2)}(G) = 0$, then $G \curvearrowright (X, \mu)$ always is the disjoint union of a probability measure preserving Bernoulli action and a dissipative action.

Theorem (Björklund-Kosloff-V, 2019)

If $b_1^{(2)}(G) > 0$, then G admits a nonsingular Bernoulli action $G \curvearrowright (X, \mu)$ that is ergodic and of type III_1.

Nonsingular Bernoulli actions of nonamenable groups

⤳ We only have results under the hypothesis that for a.e. $x \in X_0$, $\sup_{g\in G} |\log(d\mu_g/d\mu_e)(x)| < \infty$.

V-Wahl, 2017 : if $\sum_{g\in G} \exp\left(-\|c_g\|_2^2\right) < +\infty$, then $G \curvearrowright (X,\mu)$ is dissipative.

Observation : $\|c_g\|^2 \leq C(g) := \dfrac{1}{2} \sum_{h\in G} \log \int_{X_0} \dfrac{d\mu_h}{d\mu_{gh}}\, d\mu_h.$

⤳ Assume that $\sum_{g\in G} \exp\left(-7(C(g) + C(g^{-1}))\right) = +\infty$.

Theorem (Berendschot-V, 2020)

Under these assumptions, $G \curvearrowright (X,\mu)$ is ergodic, and there is a formula for the type.

It is never of type II_∞ or type III_0.

If G has only one end, $G \curvearrowright (X,\mu)$ is also not of type III_λ with $\lambda \in (0,1)$.

Open problems

$\rightsquigarrow$ An infinite group G has only one end if a subset $W \subset G$ with the property that $|gW \vartriangle W| < \infty$ for all $g \in G$, is either finite or cofinite.

- Is there a nonamenable group G that admits a Bernoulli action of type II_∞ or type III_0 ? My guess: no.
- Is there a nonamenable group G with only one end that admits a Bernoulli action of type III_λ with $\lambda \in (0,1)$? My guess: no.
- Does every nonamenable group G with more than one end admit Bernoulli actions of type III_λ for all $\lambda \in (0,1]$? My guess: I have no idea.

  Note: we proved that these groups admit Bernoulli actions of III_λ for all $\lambda \in (\lambda_0, 1]$ for some $\lambda_0 < 1$.