Spectral gap characterizations of property (T) for II_1 Factors

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COSy, 2022

Goal: characterization of Property (T) by spectral gaps in inclusions into tracial von Neumann algebras for separable II_1 factors.

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(A question asked in [Goldbring, 2020])

von Neumann Algebra Basics

- A *tracial von Neumann algebra* is a von Neumann algebra M equipped with a normal (continuous with respect to the w.o. topology on the unit ball) faithful ($\tau(x^*x) = 0$ only if $x = 0$) tracial ($\tau(xy) = \tau(yx)$ for all $x, y \in M$) state (positive linear functional, $\tau(1) = 1$).

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- A II_1 *factor* is an infinite dimensional tracial von Neumann algebra with a trivial center.

Hilbert bimodules

- For two tracial von Neumann algebras (M, τ_M) and (N, τ_N), a *M-N-bimodule* is a Hilbert space $\mathcal{H}$ equipped with a normal unital homomorphism $\pi_l : M \to \mathcal{B}(\mathcal{H})$ and a normal unital anti-homomorphism $\pi_r : N \to \mathcal{B}(\mathcal{H})$ such that π_l and π_r commute.

Bimodules (Correspondences) and Connes' Tensor Product

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Example

Given a tracial von Neumann algebra (M, τ_M), let $L^2(M, \tau)$ be the completion of M with the inner product $\langle x, y \rangle = \tau(x^* y)$. Then $L^2(M, \tau)$ is an M-M-bimodule.

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- For an M-N-bimodule ${}_M\mathcal{H}_N$ and an N-P-bimodule ${}_NK_P$, where M, N and P are three tracial von Neumann algebras, the *Connes fusion tensor product* $\mathcal{H} \otimes_N \mathcal{K}$ is a M-P-bimodule .

Kazhdan Property (T)

For groups

Let Γ be a discrete group. Then Γ has *Property (T)* ([Kazhdan, 1967]), if for any unitary representation $(\pi, \mathcal{H})$ of Γ with almost invariant unit vectors ξ_i's:

$$\xi_i \in \mathcal{H} \text{ such that } ||\pi(\gamma)\xi_i - \xi_i|| \to 0 \text{ for every } \gamma \in \Gamma,$$

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Examples

- Finite groups.
- $SL_n(\mathbb{Z})$, $n \geq 3$.

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For II_1 factors

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- The group von Neumann algebra $L(\Gamma)$ has Property (T) iff Γ has Property (T).

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(we always have the inclusion $A' \cap M^\omega \supseteq (A' \cap M)^\omega$)

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Suppose an M-M-bimodule $\mathcal{H}$ has almost central, unit vectors ξ_i but no non-zero central vectors. Construct from $\mathcal{H}$ an inclusion $M \subseteq \tilde{M}$ such that $(M' \cap \tilde{M})^\omega \subsetneq M' \cap \tilde{M}^\omega$.

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$$l(\xi)(x) = \xi x \text{ for } x \in L^2(M),$$
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- $\tilde{\mathcal{H}} \cong L^2(\tilde{M}, \tau_{\tilde{M}})$ as M-M-bimodules.

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So $2 \Leftrightarrow 1$.

Weak spectral gap only in irreducible inclusions

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Extra step. We need $\mathcal{H}$ to satisfy $\mathcal{H}^{\bigotimes_M^n}$ not to have non-zero central vectors as an M-M-bimodule, so that $M' \cap L^2(\tilde{M}) = \mathbb{C}1$.

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The following are equivalent definitions for $\mathcal{H}$ being a (left) weakly mixing M-M-bimodule:

1. the M-M-bimodule $\mathcal{H} \otimes \overline{\mathcal{H}}$ contains no non-zero central vector;
2. $\mathcal{H}$ has no non-zero right M-finite dimensional subbimodule;
3. there exists a sequence of unitaries $(u_n) \subset \mathcal{U}(M)$ such that $\lim_n \sup_{b \in (N)_1} |\langle u_n \xi b, \eta \rangle| = 0$ for any ξ and η in $\mathcal{H}$.

Characterization of Property (T) with non weak mixing

We need to show if M does not have Property (T) then there is an M-M-bimodule $\mathcal{H}$ such that $\mathcal{H}$ has almost central, unit vectors ξ_n and $\mathcal{H}$ is weakly mixing.

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In the group case:

Theorem ([Bekka and Valette, 1993])

Let G be a group. Then the following are equivalent:

1. *G has Property (T);*
2. *any unitary representation π of G on a Hilbert space which has almost invariant unit vectors has a non-zero finite dimensional subrepresentation.*

Characterization of Property (T) with non weak mixing

Theorem ([Tan, 2022])

For a II_1 factor M, the following are equivalent:

1. M has Property (T);
4. for any M-M-bimodule $\mathcal{H}$ with almost central unit vectors, $\mathcal{H}$ has a subbimodule $\mathcal{K}$ which is left or right finite M-dimensional (*not* both left and right weakly mixing);

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2. any inclusion of M into a tracial von Neumann algebra $\tilde{M}$ has weak spectral gap, i.e. $M' \cap \tilde{M}^\omega = (M' \cap \tilde{M})^\omega$;
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