Interacting Particle Systems:

Zero-Range Interaction and Properties

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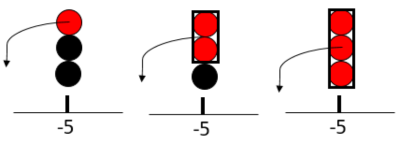
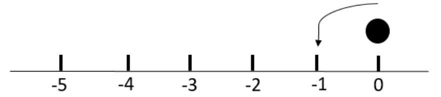


Section 1: Zero-range processes

Particle is positioned at 0 and after waiting an exponential amount of time it jumps one site to the left.

- Inter-jump time $\sim \mathrm{Exp}(\lambda)$
- Particle position at time $t \sim \mathrm{Pois}(\lambda t)$
- Expected position of the particle at time $t = \lambda t$
- Particle positions over time is a Poisson process $(N(t))_{t \geq 0}$



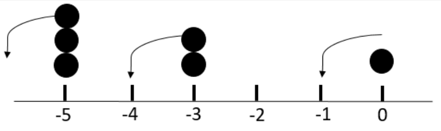
Random Walk: multiple particles

Several particles are positioned at sites of the one-dimensional lattice. Independently, each waits an exponential amount of time and then jumps one site to the left.

- Particle distribution at time 0 is a product of Poisson distributions
- Inter-jump time $\sim Exp(\lambda)$
- Particle distribution at time t is a product of Poisson distributions

Define the configuration $\eta = (\eta(x))_{x\in\mathbb{Z}}$, where $\eta(x)$ is the number of particles at site x.

$$(Lf)(\eta) = \sum_{x\in\mathbb{Z}} \eta(x)(f(\eta^{x,-}) - f(\eta))$$



TAZRP: Totally asymmetric zero-range process

A Markov jump process with generator

$$(Lf)(\eta) = \sum_{x\in\mathbb{Z}} g(\eta(x))(f(\eta^{x,-}) - f(\eta))$$

- totally: jumps occur only to the left
- asymmetric: jumps to the left are more likely to occur than jumps to the right
- zero-range: jump rate depends on the departure occupancy, only
- $g : \{0, 1, 2, \dots\} \to \mathbb{R}_+$ bounded, non-decreasing function with $g(0) = 0$

Invariant measures that are translation-invariant and product measures

$$\nu^\rho(k) = \frac{1}{Z(\lambda)} \frac{\lambda^k}{g(0)g(1)\ldots g(k)}$$

- λ is selected so that the expected value of ν^ρ is ρ
- $Z(\lambda)$ is a normalizing constant

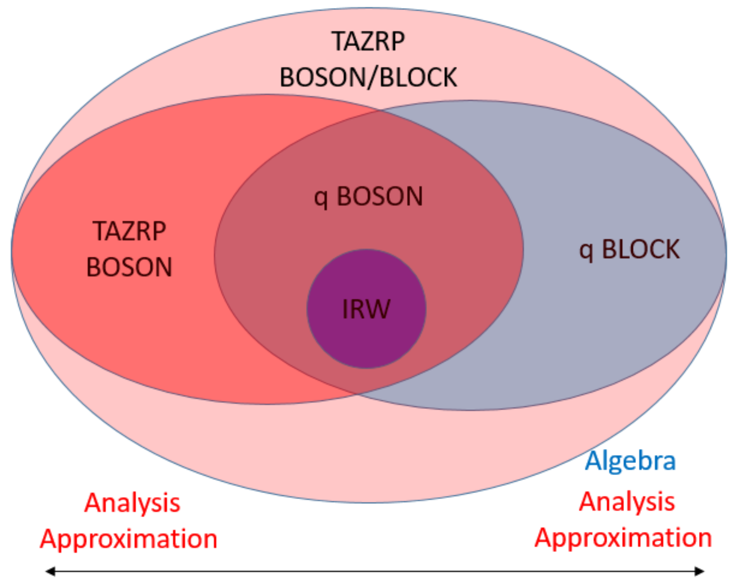
TAZRP: q-BOSON

TAZRP with $g(n) = [n] = \frac{1-q^n}{1-q}$
= q-BOSON of Sasamoto and Wadati[1]
= q-TAZRP of Borodin and Corwin [2]

[1]Sasamoto T, Wadati M, Exact results for one-dimensional totaly asymmetric diffusion models, J. Phys. A 31 (1998) 6057-6071

[2]Borodin A, Corwin I, Macdonald Processes, Probab Theor Rel Fields 158 (2014) 225-400

MODEL HIERARCHY



Section 2: Algebraic properties

Deformed affine Hecke algebra of type A_{k-1}

Generators

$$T's : T_1, \ldots, T_{k-1} \quad \text{and} \quad X's : X_1, \ldots, X_k, X_1^{-1}, \ldots X_k^{-1}$$

Eigenvalue relations

$$(T_i - 1)(T_i + q) = 0, \quad i = 1, \ldots, k$$

Braid relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i = 1, \ldots, k$$
$$T_i T_j = T_j T_i, \quad |i - j| > 1$$

Laurent relations

$$X_i X_j = X_j X_i, \quad X_i X_i^{-1} = X_i^{-1} X_i, \quad i, j = 1, \ldots, k$$

Simple action relations

$$T_i X_i T_i = q X_{i+1} \quad i = 1, \ldots, k-1$$
$$T_i X_j = X_j T_i \quad i \neq j, j-1$$

Deformed action relations

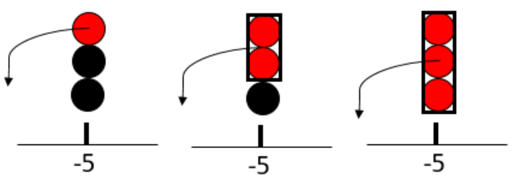
$$X_{i+1} T_i - T_i X_i = T_i X_{i+1} - T_i X_i = (\alpha + \beta X_i)(\gamma + \delta X_{i+1}) \quad i = 1, \ldots, k-1$$
$$T_i X_j = X_j T_i \quad i \neq j, j-1$$

Generators and relations of Hecke algebras

	type A_{k-1}	affine	deformed [3]
Parameters	q	q	$\alpha, \beta, \gamma, \delta$ $q = 1 + \beta\gamma - \alpha\delta$
Generators	$T's$	$T's, X's$	$T's, X's$
Eigenvalue	✓	✓	✓
Braid	✓	✓	✓
Laurent		✓	✓
Action		simple	deformed

[3] Takeyama Y, A deformation of affine Hecke algebra and integrable stochastic particle systems, J. Phys. A 47(46) (2014)

The model of Takeyama: q-BLOCK



$$H(\alpha,\beta,\gamma,\delta) = -\alpha\gamma\sum_{j=1}^{k}\frac{[d_j^+]}{1+\beta\gamma[d_j^+]}$$

$$+\sum_{r=1}^{k}(-\beta\gamma)^{r-1}[r-1]!q^{(r(r-1)/2)}$$

$$\times\sum_{1\le j_1<\cdots<j_r\le k}\frac{q^{d_{j_1}^-+\cdots+d_{j_r}^-}\delta_{j_1,\ldots,j_r}}{\prod_{p=0}^{r-1}(1+\beta\gamma[d_{j_1}^+ + d_{j_1}^- - p])}\mathbf{X_{j_1}\ldots X_{j_r}}$$

When $(\alpha+\beta)(\gamma+\delta)=0$, $H(\alpha,\beta,\gamma,\delta)$ is the generator of a Markov jump process. The dynamics involve movement to the **left** of more than one particle from one site to the neighboring site.

j particles move to the left from a cluster with a particles at

$$\text{rate} = r(a,j) = \frac{s^{j-1}}{[j]} \prod_{p=0}^{j-1} \frac{[a-p]}{1+s[a-1-p]}$$

- for $r=1$, rate $= r(a,1) = \frac{[a]}{1+s[a-1]}$
- for $r=2$, rate $= r(a,2) = \frac{s[a][a-1]}{(1+q)(1+s[a-1])(1+s[a-2])}$

Limiting cases:

- $s=0$, $0<q<1$, q-BOSON of Sasamoto and Wadati[4] [5]
- $s=0$, $q=1$, Independent Random Walks.

[4]Sasamoto T, Wadati M, Exact results for one-dimensional totaly asymmetric diffusion models, J. Phys. A 31 (1998) 6057-6071

[5]van Diejen JF, Emsiz E, Diagonalization of the infinite q-boson system, J Functional Analysis 266 (2014) 5801-5817

Properties of Takeyama's Hamiltonian

Theorem (Takeyama, 2014)

$$H(\alpha, \beta, \gamma, \delta)G_x = G_x(X_1 + \cdots + X_n) = G_x\Delta_{1/2}$$

Proposition

Define $Y_i = X^{-1}_{k+1-i}$ for $i = 1, \ldots, k$ and $S_i = T_{k-i}$ for $i = 1, \ldots, k-1$. Then $S's$ and $Y's$ satisfies the all relations of the deformed affine Hecke algebra with parameters $\delta, \gamma, \beta, \alpha$.

$$X^{-1}_{i+1}(X_{i+1}T_i - T_iX_i)X_i^{-1} = T_iX_i^{-1} - X^{-1}_{i+1}T_i = (\delta + \gamma X^{-1}_{i+1})(\beta + \alpha X_i^{-1})$$

$$X_i^{-1}(T_iX_{i+1} - X_iT_i)X^{-1}_{i+1} = X_i^{-1}T_i - T_iX^{-1}_{i+1} = (\delta + \gamma X^{-1}_{i+1})(\beta + \alpha X_i^{-1})$$

Consequence

- $H(\delta,\gamma,\beta,\alpha)G_y = G_y(Y_1 + \cdots + Y_n)$

$$= G_y(X_n^{-1} + \cdots + X_1^{-1}) = G_y\Delta_{-1/2}$$

- $(H(\alpha,\beta,\gamma,\delta) + H(\delta,\gamma,\beta,\alpha))G$

$$= G(X_1 + \cdots + X_n + X_1^{-1} + \cdots + X_n^{-1}) = G\Delta$$

$H(\alpha,\beta,\gamma,\delta) + H(\delta,\gamma,\beta,\alpha)$ encodes particle movement to both **left** and **right** (AZRP).

Consequence: Eigenvectors of the Hamiltonian can be constructed from the eigenvectors of the Laplacian. The Laplacian's eigenvectors are calculated via Bethe Ansatz. Knowledge of these eigenvectors help calculate transition probabilities.
For q-BOSON transition probabilities can be calculated [6] [7]

[6] Borodin A, Corwin I, Petrov L, and Sasamoto T, SPECTRAL THEORY for the q-BOSON PARTICLE SYSTEM

[7] Korhonen M and Lee E, The transition probability and the probability for the left-most particle's position of the q-TAZRP

Section 3: Analytic/Asymptotic properties

Hydrodynamic Scaling Limit

Microscopic
empirical distribution of particles
at time t=0

$$\rho^N(0) = \frac{1}{N}\sum_{x \in Z} \theta_0^N(x)\delta_{x/N}$$

$N \to \infty$

Macroscopic profile
at time t=0

$\rho_0(\cdot)$

Hamiltonian
Generator of Markov
jump process

Quasilinear PDE

$$\partial_t \rho - \partial_x\big(J(\rho)\big) = 0$$
$$\rho(0, x) = \rho_0(x)$$

Microscopic
empirical distribution of particles
at time t

$$\rho^N(t) = \frac{1}{N}\sum_{x \in Z} \theta_t^N(x)\delta_{x/N}$$

$N \to \infty$

Macroscopic profile
at time t

$\rho(t,\cdot)$

Theorem (Rezakhanlou[8] , 1991)

TAZRP has a hydrodynamic scaling limit given by the solution of the quasi-linear hyperbolic equation of first order

$$\partial_t \rho = \partial_x (J(\rho))$$

$$\rho(0, x) = \rho_0(x)$$

where $J(\rho) = E_{\nu^\rho}[g]$ is the expected microscopic current through a site

[8]Rezakhanlou F, Hydrodynamic Limit for Attractive Particle Systems on Z^d, Commun. Math. Phys. 140, 417-448 (1991)

Initial particle distribution $\rho(0,x) = \begin{cases} b & x < 0 \\ a & x \geq 0 \end{cases}$

Independent Random Walks

$$\partial_t(\rho) - \partial_x(\rho) = 0$$

$$\rho(t,x) =$$

$$\begin{cases} b & x < -t \\ a & x \geq t \end{cases}$$

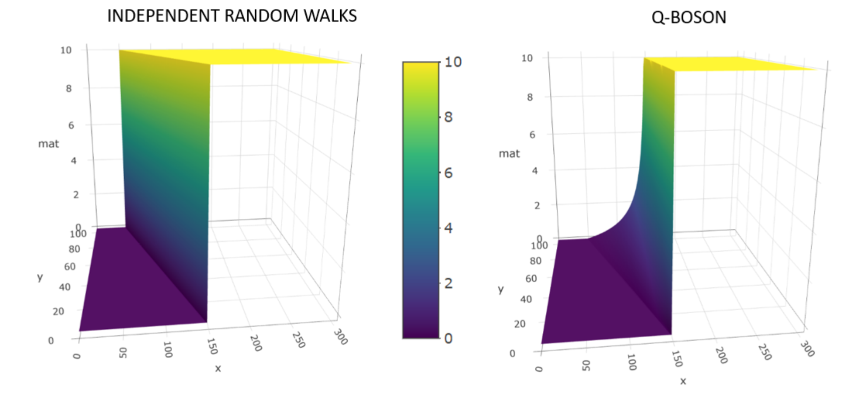
q-Boson

$$\partial_t(\rho) - \frac{1}{(1+(1-q)\rho)^2}\partial_x(\rho) = 0$$

$$\rho(t,x) =$$

$$\begin{cases} b & x < -f(t,b) \\ \frac{1}{1-q}\left(\sqrt{\frac{t}{-x}} - 1\right) & -f(t,b) \leq x < -f(t,a) \\ a & x \geq -f(t,a) \end{cases}$$

TAZRP: Hydrodynamic Scaling Limit



Proposition (Savu)

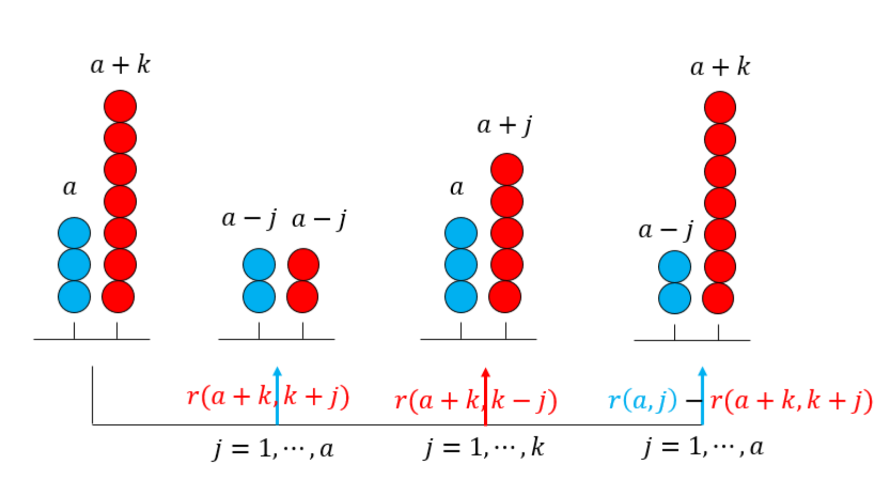
For $s \leq q$, the q-BLOCK is attractive

$$s[b+1] = s(1+q+\cdots+q^b) = s(1+q[b]) \leq q + sq[b] = q(1+s[b])$$

$\Rightarrow$

$$r(a+k, j+k) = r(a,j) \times \frac{[j]}{[j+k]} \times \frac{s[a+k]}{1+s[a+k-1]} \times \cdots \times \frac{s[a+1]}{1+s[a]}$$

$$\leq r(a,j) \times \frac{q^k[j]}{[j+k]} \leq r(a,j)$$



$a+k$
a
$a-j$ $a-j$
$a+j$
a
$a+k$
$a-j$
$r(a+k,k+j)$
$r(a+k,k-j)$
$r(a,j)-r(a+k,k+j)$
$j=1,\cdots,a$
$j=1,\cdots,k$
$j=1,\cdots,a$

Theorem (in progress)

For $s \leq q$, the q-BLOCK has a hydrodynamic scaling limit given by the solution of the quasi-linear hyperbolic equation of first order

$$\partial_t \rho = \partial_x (J(\rho))$$

$$\rho(0, x) = \rho_0(x)$$

where $J(\rho) = E_{\nu^\rho(da)}\left[\sum_{j=1}^{a} j r(a, j) \right]$ is the expected microscopic current through a site.

q-BLOCK: Kolmogorov Forward Equation 2P

$$\frac{dP_t}{dt}(y,x) = P_t(y+1,x) + P_t(y,x+1) - 2P_t(y,x), \quad {\scriptstyle y \geq x+2}$$

$$\frac{dP_t}{dt}(x+1,x) = P_t(x+2,x) + uP_t(x+1,x+1) - 2P_t(x+1,x)$$

$$\frac{dP_t}{dt}(x,x) = P_t(x+1,x) + vP_t(x+1,x+1) - (u+v)P_t(x,x)$$

$$u = r(2,1) = \frac{1+q}{1+s} = (1+q)(1-\lambda)$$

$$v = r(2,2) = \frac{s}{1+s} = \lambda$$

Let P_0 be the solution

$$\frac{dP_t^0}{dt}(y,x) = P_t^0(y+1,x) + P_t^0(y,x+1) - 2P_t^0(y,x), \quad {\scriptstyle y,x\in\mathbb{Z}}$$

satisfying the boundary condition

$$P_t^0(x,x+1) = (u-1)P_t^0(x+1,x) + vP_t^0(x+1,x+1) - (u+v-2)P_t^0(x,x)$$

Then P_t satisfies KF equation for 2 particles

$$P_t(y,x) = \begin{cases} P_t^0(y,x), & \text{for} \quad y \geq x+1 \\ \frac{1}{u}P_t^0(x,x), & \text{for} \quad y = x \end{cases}$$

Weighting $\rightarrow$ RED EQ.
Weighting + Boundary Condition $\rightarrow$ BLUE EQ.

q-BLOCK: Kolmogorov Forward Equation 3P

$$\frac{dP_t}{dt}(z,y,x)=P_t(z+1,y,x)+P_t(z,y+1,x)+P_t(z,y,x+1)-3P_t(z,y,x), \quad z\geq y+2\geq x+4$$

$$\frac{dP_t}{dt}(z,x+1,x)=P_t(z+1,x+1,x)+P_t(z,x+2,x+1)+r_1^2P_t(z,x+1,x+1)-3P_t(z,x+1,x), \quad z\geq x+3$$

$$\frac{dP_t}{dt}(x+1,x,y)=P_t(x+2,x,y)+r_1^2P_t(x+1,x+1,y)+P_t(x+1,x,y+1)-3P_t(x+1,x,y), \quad x\geq y+2$$

$$\frac{dP_t}{dt}(x+1,x,x-1)=P_t(x+2,x,x-1)+r_1^2P_t(x+1,x+1,x-1)+r_1^2P_t(x+1,x,x)-3P_t(x+1,x,x-1)$$

$$\frac{dP_t}{dt}(z,x,x)=P_t(z+1,x,x)+P_t(z,x+1,x)+r_2^2P_t(z,x+1,x+1)-(1+r_1^2+r_2^2)P_t(z,x,x), \quad z\geq x+2$$

$$\frac{dP_t}{dt}(x,x,y)=P_t(x+1,x,y)+r_2^2P_t(x+1,x+1,y)+P_t(x,x,y+1)-(1+r_1^2+r_2^2)P_t(x,x,y), \quad x\geq y+2$$

$$\frac{dP_t}{dt}(x+1,x,x)=P_t(x+2,x,x)+r_1^2P_t(x+1,x+1,x)+r_2^3P_t(x+1,x+1,x+1)-(1+r_1^2+r_2^2)P_t(x+1,x,x)$$

$$\frac{dP_t}{dt}(x,x,x-1)=P_t(x+1,x,x-1)+r_2^2P_t(x+1,x+1,x-1)+r_1^3P_t(x,x,x)-(1+r_1^2+r_2^2)P_t(x,x,x-1)$$

$$\frac{dP_t}{dt}(x,x,x)=P_t(x+1,x,x)+r_2^2P_t(x+1,x+1,x)+r_3^3P_t(x+1,x+1,x+1)-(r_1^3+r_2^3+r_3^3)P_t(x,x,x)$$

Let P_0 be the solution

$$\frac{dP_t^0}{dt}(z,y,x)=P_t^0(z+1,y,x)+P_t^0(z,y+1,x)+P_t^0(z,y,x+1)-3P_t^0(z,y,x), \quad z,y,x\in\mathbb{Z}$$

satisfying the boundary condition

$$\begin{aligned} P_t^0(x,x+1,y) &= (r_1^2-1)P_t^0(x+1,x,y)+r_2^2P_t^0(x+1,x+1,y)+(2-r_1^2+r_2^2)P_t^0(x,x,y) \quad x+1\geq y \\ P_t^0(z,x,x+1) &= (r_1^2-1)P_t^0(z,x+1,x)+r_2^2P_t^0(z,x+1,x+1)+(2-r_1^2+r_2^2)P_t^0(z,x,x) \quad z\geq x+1 \end{aligned}$$

Then P_t satisfies KF equation for 3 particles

$$P_t(z,y,x) = \begin{cases} P_t^0(z,y,x), & \text{for} \quad z\geq y+1\geq x+2 \\ \frac{1}{r(2,1)}P_t^0(z,y,x), & \text{for} \quad z=y\geq x+1 \text{ or} \\ & \qquad\quad z-1\geq y=x \\ \frac{1}{r(2,1)r(3,1)}P_t^0(z,y,x), & \text{for} \quad x=y=z \end{cases}$$

under the constraints on the transition rates

$$\begin{aligned} \textit{Constrain } 1 \quad r(3,1) &= 1 + \frac{r(2,1)(r(2,1)-1)}{1-r(2,2)(2-r(2,1)-r(2,2))} \\ \textit{Constrain } 2 \quad r(3,2) &= r(3,1)r(2,1) \\ \textit{Constrain } 3 \quad r(3,3) &= \frac{r(2,1)r(2,2)^2}{1-r(2,2)(2-r(2,1)-r(2,2))} \end{aligned}$$

Weighting $\rightarrow$ RED EQ.
Weighting + Boundary Condition $\rightarrow$ BLUE EQ.
Weighting + Boundary Condition + Constrain 2 $\rightarrow$ GREEN EQ.
Weighting + Boundary Condition + Constrains 2 + Constrains 1, 3 $\rightarrow$ ORANGE EQ.