Irreducible inclusions of simple C^*-algebras

Mikael Rørdam
University of Copenhagen

Canadian Operator Algebras Symposium
June 2, 2022

Outline

1. C^*-irreducible inclusions
2. von Neumann algebras
3. Inductive limits and tensor products
4. Inclusions arising from groups and dynamics
5. Bisch–Haagerup type inclusions

Definition: A unital inclusion $\mathcal{B} \subseteq \mathcal{A}$ of unital (simple) C^*-algebra is *C^*-irreducible* if all intermediate C^*-algebras $\mathcal{B} \subseteq \mathcal{D} \subseteq \mathcal{A}$ are simple.

Example [Kishimoto, Olesen–Pedersen]: Let $\Gamma \curvearrowright \mathcal{B}$ be an outer action of a discrete group Γ on a unital simple C^*-algebra $\mathcal{B}$.

Then $\mathcal{B} \subseteq \mathcal{B} \rtimes_{\mathrm{red}} \Gamma$ is C^*-irreducible.

von Neumann analogy: For an inclusion $\mathcal{N} \subseteq \mathcal{M}$ of vNalg TFAE:

1. $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$,
2. all intermediate von Neumann algs $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{M}$ are factors,
3. $\bigvee_{u \in \mathcal{U}(\mathcal{N})} upu^* = 1$, for all non-zero projections $p \in \mathcal{M}$.

Goals:

- Give intrinsic characterization of C^*-irreducible inclusions.
- Provide (further) examples and derive properties of C^*-irreducible inclusions.
- Rigidity: classify intermediate C^*-algebras of an C^*-irreducible inclusions.

Full elements: $\mathcal{A}$ = unital C^*-alg and $a \in \mathcal{A}^+$. Then a is full in $\mathcal{A}$ (i.e., not contained in any proper two-sided ideal) iff $\exists x_1, \ldots, x_n \in \mathcal{A}$ st $\sum_{j=1}^n x_j^* a x_j \geq 1_{\mathcal{A}}$.

Lemma: $\mathcal{A}$ = unital C^*-alg, $\mathcal{W} \subseteq \mathcal{A}$, and $a \in \mathcal{A}^+$.

$$\exists x_1, \ldots, x_n \in \overline{\mathrm{span}(\mathcal{W})} \text{ st } \sum_{j=1}^n x_j^* a x_j \geq 1_{\mathcal{A}}$$
$$\implies \exists w_1, \ldots, w_m \in \mathcal{W} \text{ st } \sum_{j=1}^m w_j^* a w_j \geq 1_{\mathcal{A}}.$$

Def./Lemma: Given unital inclusion $\mathcal{B} \subseteq \mathcal{A}$ of C^*-algs, and $a \in \mathcal{A}^+$. a is full relatively to $\mathcal{B}$ if one of the following equivalent conditions hold:

- $\exists x_1, \ldots, x_n \in \mathcal{B}$ st $\sum_{j=1}^n x_j^* a x_j \geq 1_{\mathcal{A}}$.
- $\exists x_1, \ldots, x_n \in \mathcal{B}$ st $\sum_{j=1}^n x_j^* a x_j$ invertible,
- $\exists u_1, \ldots, u_m \in \mathcal{U}(\mathcal{B})$ st $\sum_{j=1}^m u_j^* a u_j \geq 1_{\mathcal{A}}$.

Proposition: Given unital inclusion $\mathcal{B} \subseteq \mathcal{A}$ of C^*-algs, and $a \in \mathcal{A}^+$. Then: a is full relatively to $\mathcal{B}$ $\iff$ a is full in $C^*(\mathcal{B}, a)$.

Proposition: Given unital inclusion $\mathcal{B} \subseteq \mathcal{A}$ of C^*-algs, and $a \in \mathcal{A}^+$.
Then: a is full relatively to $\mathcal{B} \iff a$ is full in $C^*(\mathcal{B}, a)$.

Proof: Apply the lemma to $\mathcal{W} = \{b_1 a b_2 a \cdots a b_n : n \geq 1, b_j \in \mathcal{B}\}$.

Theorem: A unital inclusion $\mathcal{B} \subseteq \mathcal{A}$ of C^*-algebras is C^*-irreducible (all intermediate C^*-algs are simple) iff each non-zero $a \in \mathcal{A}^+$ is full rel. to $\mathcal{B}$.

Proof: "only if". Let $a \in \mathcal{A}^+$ be non-zero. Then $\mathcal{B} \subseteq C^*(\mathcal{B}, a) \subseteq \mathcal{A}$, and

$$C^*(\mathcal{B}, a) \text{ simple} \Rightarrow a \text{ full in } C^*(\mathcal{B}, a) \Leftrightarrow a \text{ full relatively to } \mathcal{B}.$$

Fact: $\mathcal{B} \subseteq \mathcal{A}$ is C^*-irreducible $\Rightarrow \mathcal{B} \subseteq \mathcal{A}$ is irreducible (i.e., $\mathcal{B}' \cap \mathcal{A} = \mathbb{C}$).

Proof: $a \in (\mathcal{B}' \cap \mathcal{A})^+$ non-invertible $\implies a$ not full rel. to $\mathcal{B}$.

- $\mathcal{B}' \cap \mathcal{A} = \mathbb{C}$ and $\mathcal{A}$, $\mathcal{B}$ simple unital $\not\Rightarrow \mathcal{B} \subseteq \mathcal{A}$ is C^*-irreducible.

Definition: Given unital inclusion $\mathcal{B} \subseteq \mathcal{A}$ of C^*-algs, and $a \in \mathcal{A}$. Set

$$C_{\mathcal{B}}(a) = \overline{\mathrm{conv}\{u^* a u : u \in \mathcal{U}(\mathcal{B})\}}.$$

- $\mathcal{A}$ has *Dixmier property* if $C_{\mathcal{A}}(a) \cap \mathbb{C}1_{\mathcal{A}} \neq \emptyset \ \forall a \in \mathcal{A}$
- $\mathcal{B} \subseteq \mathcal{A}$ has the *relative Dixmier property* if $C_{\mathcal{B}}(a) \cap \mathbb{C}1_{\mathcal{A}} \neq \emptyset \ \forall a \in \mathcal{A}$

Theorem [Popa]: $\mathcal{B} \subseteq \mathcal{A}$ has relative Dixmier property if

- $\mathcal{B}$ has Dixmier property,
- $\mathcal{B} \subseteq \mathcal{A}$ has finite Jones index wrt some cond. expect. $E \colon \mathcal{A} \to \mathcal{B}$,
- $\pi_\varphi(\mathcal{B})' \cap \pi_\varphi(\mathcal{A})'' = \mathbb{C}$, for some state φ on $\mathcal{A}$.

- If $\mathcal{B} \subseteq \mathcal{A}$ has the rel. Diximier property, and $\tau =$ trace on $\mathcal{A}$, then $C_{\mathcal{B}}(a) \cap \mathbb{C}1_{\mathcal{A}} = \{\tau(a) \cdot 1_{\mathcal{A}}\}$, for $a \in \mathcal{A}$.
- For $a \in \mathcal{A}$: $C_{\mathcal{B}}(a) \in \mathbb{C} \setminus \{0\} \implies a$ is full rel. to $\mathcal{B}$.

$\mathcal{B} \subseteq \mathcal{A}$ has rel. Dixmier property and $\mathcal{A}$ has (faithful) tracial state $\implies \mathcal{B} \subseteq \mathcal{A}$ is C^*-irreducible.

Definition: An inclusion $\mathcal{B} \subseteq \mathcal{A}$ with cond. expectation $E\colon \mathcal{A} \to \mathcal{B}$ has the *pinching property* if $\forall a \in \mathcal{A}^+$ and $\forall \varepsilon > 0$ $\exists$ contraction $h \in \mathcal{B}$ st

$$\|h^*(a - E(a))h\| \leq \varepsilon \qquad \text{and} \qquad \|h^* E(a)h\| \geq \|E(a)\| - \varepsilon.$$

Example [Elliott, Kishimoto, Olesen–Pedersen]: If $\Gamma \curvearrowright \mathcal{A}$ is a properly outer action of Γ on C^*-alg $\mathcal{A}$, then $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\mathrm{red}} \Gamma$ has pinching property wrt canonical cond. expectation $E\colon \mathcal{A} \rtimes_{\mathrm{red}} \Gamma \to \mathcal{A}$.

If $\mathcal{B} \subseteq \mathcal{A}$ has pinching property wrt some *faithful* cond. expectation $E\colon \mathcal{A} \to \mathcal{B}$, and $\mathcal{B}$ simple, then $\mathcal{B} \subseteq \mathcal{A}$ is C^*-irreducible.

Definition: Given inclusion $\mathcal{B} \subseteq \mathcal{A}$ of C^*-algs and cond. expect. $E \colon \mathcal{A} \to \mathcal{B}$, set

$$\mathrm{Ind}(E) = \lambda^{-1}, \qquad \lambda = \sup\{t \geq 0 \mid \forall a \in \mathcal{A}^+ : E(a) \geq ta\}.$$

Theorem [Izumi, 2002]: Given $\mathcal{B} \subseteq \mathcal{A}$ and cond. expect. $E \colon \mathcal{A} \to \mathcal{B}$ with $\mathrm{Ind}(E) < \infty$.

- If $\mathcal{A}$ (or $\mathcal{B}$) is simple, then $\mathcal{B}$ (or $\mathcal{A}$) is a finite direct sum of simple C^*-algebras.
- In particular, if $\mathcal{A} \cap \mathcal{B}' = \mathbb{C}$, then $\mathcal{A}$ is simple iff $\mathcal{B}$ is simple.

Corollary: Given $\mathcal{B} \subseteq \mathcal{A}$ simple with cond. expect. $E \colon \mathcal{A} \to \mathcal{B}$ st $\mathrm{Ind}(E) < \infty$. Then: $\mathcal{B} \subseteq \mathcal{A}$ is C^*-irreducible $\iff$ $\mathcal{A} \cap \mathcal{B}' = \mathbb{C}$.

Outline

1. C^*-irreducible inclusions
2. von Neumann algebras
3. Inductive limits and tensor products
4. Inclusions arising from groups and dynamics
5. Bisch–Haagerup type inclusions

Theorem [Popa]: Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of separable II_1-factors. TFAE:

(i) $\mathcal{N} \subseteq \mathcal{M}$ is C^*-irreducible,

(ii) $\mathcal{N} \subseteq \mathcal{M}$ has the relative Dixmier property,

(iii) $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$ and $[\mathcal{M} : \mathcal{N}] < \infty$.

(ii) $\iff$ (iii) is the main result of a paper of Popa.

(ii) $\implies$ (i) already noted.

(i) $\implies$ $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$ also already noted.

(i) $\implies$ $[\mathcal{M} : \mathcal{N}] < \infty$ follows from results of Popa, resp., F. Pop:

Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of factors.

- $\mathcal{N}' \cap \mathcal{M} = \mathbb{C}$ iff $\forall$ non-zero projections $p \in \mathcal{M}$: $\bigvee_{u \in \mathcal{U}(\mathcal{N})} u^* p u = 1$,
- $\mathcal{N} \subseteq \mathcal{M}$ is C^*-irreducible iff $\forall$ non-zero projections $p \in \mathcal{M}$ $\exists u_1, \dots, u_n \in \mathcal{U}(\mathcal{N})$ st $\sum_j u_j^* p u_j \geq 1$.

The two conditions above are not equivalent by Popa's theorem. Is the following intermediate property equivalent to any of the two above?

- $\forall$ $p \in \mathcal{M}$ non-zero projection $\exists u_1, \dots, u_n \in \mathcal{U}(\mathcal{N})$ st $\bigvee_j u_j^* p u_j = 1$.

Question: Are all irreducible inclusions $\mathcal{N} \subseteq \mathcal{M}$ of type III factors C^*-irreducible?

Outline

1. C^*-irreducible inclusions
2. von Neumann algebras
3. Inductive limits and tensor products
4. Inclusions arising from groups and dynamics
5. Bisch–Haagerup type inclusions

Goal: Build a C^*-irreducible inclusions $\mathcal{B} \subseteq \mathcal{A}$ from inductive limits:

$$\begin{array}{ccccccccc}
\mathcal{B}_1 & \xrightarrow{\mu_1} & \mathcal{B}_2 & \xrightarrow{\mu_2} & \mathcal{B}_3 & \xrightarrow{\mu_3} & \cdots & \longrightarrow & \mathcal{B} \\
\downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_3} & & & & \downarrow{\scriptstyle \iota} \\
\mathcal{A}_1 & \xrightarrow{\lambda_1} & \mathcal{A}_2 & \xrightarrow{\lambda_2} & \mathcal{A}_3 & \xrightarrow{\lambda_3} & \cdots & \longrightarrow & \mathcal{A}
\end{array}$$

Proposition: $\mathcal{B} \subseteq \mathcal{A}$ is C^*-irreducible iff $\forall n \geq 1$ $\forall a \in \mathcal{A}_n^+ \setminus \{0\}$ $\exists\, m \geq n$ st $\lambda_{m,n}(a)$ is full relatively to $\iota_m(\mathcal{B}_m)$.

Proposition: $\mathcal{A}, \mathcal{B} \subseteq \mathcal{E}$ unital fin. dim. C^*-algs. Every $a \in \mathcal{A}^+ \setminus \{0\}$ is full relatively to $\mathcal{B}$ $\iff$ $\mathcal{A}$ and $\mathcal{B}' \cap \mathcal{E}$ are everywhere non-othogonal in $\mathcal{E}$, i.e.,

$$\forall x \in \mathcal{A}^+ \quad \forall y \in (\mathcal{B}' \cap \mathcal{E})^+ : \quad x \perp y \implies x = 0 \text{ or } y = 0.$$

Corollary: Given inductive systems of inclusions as above with all $\mathcal{B}_n$ and $\mathcal{A}_n$ finite dimensional. Then $\mathcal{B} \subseteq \mathcal{A}$ is C^*-irreducible if $\forall n \geq 1$:

$$\lambda_n(\mathcal{A}_n) \subseteq \mathcal{A}_{n+1} \text{ and } \iota_{n+1}(\mathcal{B}_{n+1})' \cap \mathcal{A}_{n+1} \subseteq \mathcal{A}_{n+1}$$

are everywhere non-orthogonal.

Combining the facts above one can construct C^*-irreducible inclusions of UHF-algebras (and AF-algebras), eg:

Corollary: If $\mathcal{B}_n \subseteq \mathcal{A}_n$ is C^*-irreducible for all n, then $\mathcal{B} \subseteq \mathcal{A}$ is C^*-irreducible as well.

Theorem: For each pair of UHF-algebras $\mathcal{A}$ and $\mathcal{B}$ for which there exists a unital inclusion $\mathcal{B} \to \mathcal{A}$, there also exists a C^*-irreducible inclusion $\mathcal{B} \to \mathcal{A}$.

- I expect that this theorem also holds with UHF-algebras replaced with simple AF-algebras.

Example: One can also obtain non-trivial C^*-irreducible of UHF-algebras as $\mathcal{B} \subseteq \mathcal{B} \rtimes \mathbb{Z}_d$, for some outer action $\mathbb{Z}_d \curvearrowright \mathcal{B}$ on a UHF-alg $\mathcal{B}$

Warning: One cannot detect C^*-irreducibilty of an inclusion $\mathcal{B} \subseteq \mathcal{A}$ of AF-algebras by K_0 or by their Bratteli diagrams.

Theorem (Zacharias–Zsido):
$\mathcal{B} =$ any unital C^*-alg.
$\mathcal{E} =$ unital simple C^*-alg with Wassermann's property (S),

Then each intermediate C^*-alg $\mathcal{E} \otimes \mathbb{C} \subseteq \mathcal{D} \subseteq \mathcal{E} \otimes \mathcal{B}$ is of the form $\mathcal{D} = \mathcal{E} \otimes \mathcal{B}_0$, for some $\mathcal{B}_0 \subseteq \mathcal{B}$.

Corollary: If $\mathcal{B} \subseteq \mathcal{A}$ is C^*-irreducible and $\mathcal{E}$ is as above (e.g., $\mathcal{E} = \mathcal{Z}$ or $\mathcal{E} = \mathcal{O}_\infty$), then $\mathcal{E} \otimes \mathcal{B} \subseteq \mathcal{E} \otimes \mathcal{A}$ is C^*-irreducible as well.

Question: Let $\mathcal{B}_i \subseteq \mathcal{A}_i$, $i \in I$, be C^*-irreducible. Is it true that

$$\bigotimes_{i \in I} \mathcal{B}_i \subseteq \bigotimes_{i \in I} \mathcal{A}_i$$

is C^*-irreducible? It suffices to check this when $|I| = 2$.

Yes, if each $\mathcal{B}_i \subseteq \mathcal{A}_i$ has relative Dixmier property or pinching property.

Outline

1. C^*-irreducible inclusions
2. von Neumann algebras
3. Inductive limits and tensor products
4. Inclusions arising from groups and dynamics
5. Bisch–Haagerup type inclusions

Given an action $\Gamma \curvearrowright \mathcal{A}$ of a group Γ on a unital C^*-alg $\mathcal{A}$.

- $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\rm red} \Gamma$,
- $C^*_\lambda(\Gamma) \subseteq \mathcal{A} \rtimes_{\rm red} \Gamma$.

The former inclusion is well understood:

Theorem [Kishimoto, Olesen–Pedersen, Popa]: TFAE when $\mathcal{A}$ simple:

(i) $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\rm red} \Gamma$ is C^*-irreducible,

(ii) $\Gamma \curvearrowright \mathcal{A}$ is outer,

(iii) $\mathcal{A}' \cap (\mathcal{A} \rtimes_{\rm red} \Gamma) = \mathbb{C}$.

Moreover, if $\mathcal{A}$ has the Dixmier property (i.e., has at most tracial state), then (i)–(iii) are equivalent to:

(iv) $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\rm red} \Gamma$ has the relative Dixmier property.

(ii) $\Rightarrow$ $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\rm red} \Gamma$ has pinching property wrt $E \colon \mathcal{A} \rtimes_{\rm red} \Gamma \to \mathcal{A}$ $\Rightarrow$ (i).

(ii) $\Rightarrow$ (iv) by a result of Popa.

Theorem [Cameron–Smith, 2019; Izumi 2002]: If $\Gamma \curvearrowright \mathcal{A}$ is an outer action on a unital simple C^*-alg $\mathcal{A}$ (so that $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\mathrm{red}} \Gamma$ is C^*-irreducible), then

$$\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{A} \rtimes_{\mathrm{red}} \Gamma \implies \mathcal{D} = \mathcal{A} \rtimes_{\mathrm{red}} \Lambda,$$

for some $\Lambda \subseteq \Gamma$.

Note to thm: For $x = \sum_{t \in \Gamma} a_t u_t \in \mathcal{A} \rtimes_{\mathrm{red}} \Gamma$ (with $a_t \in \mathcal{A}$ and $t \mapsto u_t$ repn of Γ in $\mathcal{A} \rtimes_{\mathrm{red}} \Gamma$), set

$$\mathrm{supp}(x) = \{t \in \Gamma : a_t \neq 0\}.$$

Must show: $t \in \mathrm{supp}(x) \Rightarrow u_t \in C^*(\mathcal{A}, x)$. This can quite easily be done using the following:

Lemma (Popa): $\mathcal{A} =$ simple C^*-alg w/ Dixmier property, $a_1, \dots, a_n \in \mathcal{A}$, $\alpha_1, \dots, \alpha_n \in \mathrm{Aut}(\mathcal{A})$ outer, $\varepsilon > 0$. Then $\exists v_1, \dots, v_m \in \mathcal{U}(\mathcal{A})$ st

$$\Big\| \frac{1}{m} \sum_{j=1}^{m} v_j a_i \alpha_i(v_j)^* \Big\| < \varepsilon, \quad i = 1, \dots, n.$$

Bedos–Omland extended both theorems above to the case of inclusions $\mathcal{A} \rtimes \Lambda \subseteq \mathcal{A} \rtimes \Gamma$ where $\Lambda \lhd \Gamma$ is *normal*: If $\mathcal{A} \rtimes \Lambda$ is simple, then

$\mathcal{A} \rtimes \Lambda \subseteq \mathcal{A} \rtimes \Gamma$ is C^*-irreducible $\iff$ $\mathcal{A} \rtimes \Lambda \subseteq \mathcal{A} \rtimes \Gamma$ is irreducible $\iff$ $\Gamma/\Lambda \curvearrowright \mathcal{A} \rtimes \Lambda$ is outer.

Under these conditions they further show that all intermediate C^*-algs arise as crossed products wrt intermediate groups.

Example: For $2 \leq n < \infty$ the inclusion $M_{n^\infty} \subseteq \mathcal{O}_n$ is C^*-irreducible because it satisfies the pinching property (as shown by Cuntz).

This inclusion also satisfies the averaging lemma by Popa, and hence also an analog of the Cameron–Smith–Izumi thm:

$$M_{n^\infty} \subseteq \mathcal{D} \subseteq \mathcal{O}_n \implies \mathcal{D} = C^*(M_{n^\infty}, s_1^k), \text{ for some } k \geq 0.$$

Theorem: Given $\alpha\colon \Gamma \curvearrowright \mathcal{A}$ where $\Gamma = C^*$-simple group and $\mathcal{A} =$ unital C^*-alg. TFAE:

(i) $C^*_\lambda(\Gamma) \subseteq \mathcal{A} \rtimes_{\mathrm{red}} \Gamma$ is C^*-irreducible,

(ii) $\forall a \in \mathcal{A}^+$, $a \neq 0$, $\exists t_1, \dots, t_n \in \Gamma$ st $\sum_{j=1}^n \alpha_{t_j}(a) \geq 1_{\mathcal{A}}$,

(iii) each state ϕ on $\mathcal{A}$ is Γ-*faithful*: $a \in \mathcal{A}^+$:
$\forall t \in \Gamma : \phi(\alpha_t(a)) = 0 \implies a = 0$,

(iv) $\exists\, \mu \in \mathrm{Prob}(\Gamma)$ st each μ-stationary state ϕ on $\mathcal{A}$ is faithful.

Amrutam–Kalantar proved (iv) $\Rightarrow$ (i) (assuming μ in (iv) is C^*-simple).

- Condition (ii) says that the action α is "strongly mixing" and implies minimality.
- For $\mathcal{A}$ commutative: (ii) $\iff$ α is minimal.
- BKKO: Γ C^*-simple and $\Gamma \curvearrowright \mathcal{A}$ minimal $\implies$ $\mathcal{A} \rtimes_{\mathrm{red}} \Gamma$ simple.

Theorem: Given $\alpha\colon \Gamma \curvearrowright \mathcal{A}$ where $\Gamma = C^*$-simple group and $\mathcal{A}$ = unital C^*-alg. TFAE:

(i) $C^*_\lambda(\Gamma) \subseteq \mathcal{A} \rtimes_{\mathrm{red}} \Gamma$ is C^*-irreducible,

(ii) $\forall a \in \mathcal{A}^+$, $a \neq 0$, $\exists t_1, \dots, t_n \in \Gamma$ st $\sum_{j=1}^n \alpha_{t_j}(a) \geq 1_\mathcal{A}$,

(iii) each state ϕ on $\mathcal{A}$ is Γ-*faithful*: $a \in \mathcal{A}^+$:
$\forall t \in \Gamma : \phi(\alpha_t(a)) = 0 \implies a = 0$,

(iv) $\exists\, \mu \in \mathrm{Prob}(\Gamma)$ st each μ-stationary state ϕ on $\mathcal{A}$ is faithful.

Theorem [Amrutam–Ursu, 2021]: If we further assume Γ has (AP) and $\mathrm{Ker}(\alpha\colon \Gamma \to \mathrm{Aut}(\mathcal{A})) \subseteq \Gamma$ is "plump", then

$$C^*_\lambda(\Gamma) \subseteq \mathcal{D} \subseteq \mathcal{A} \rtimes_{\mathrm{red}} \Gamma \implies \mathcal{D} = \mathcal{B} \rtimes \Gamma,$$

for some Γ-invariant $\mathcal{B} \subseteq \mathcal{A}$.

Remark: $\exists$ irreducible inclusions $C^*_\lambda(\Gamma) \subseteq \mathcal{A} \rtimes_{\mathrm{red}} \Gamma$ of simple C^*-algs that are not C^*-irreducible.

We now consider inclusions of C^*-algs (and von Neumann algs) arising from inclusions $\Lambda \subseteq \Gamma$ of groups.

Definition: Γ is icc rel. to Λ iff $\{tst^{-1} : t \in \Lambda\}$ is infinite $\forall s \in \Gamma \setminus \{e\}$.

Proposition: Given groups $\Lambda \subseteq \Gamma$. Then
$\mathcal{L}(\Lambda) \subseteq \mathcal{L}(\Gamma)$ is C^*-irreducibe $\iff$ Γ is icc rel. to Λ and $[\Gamma : \Lambda] < \infty$.

Proof:

- $\mathcal{L}(\Lambda)' \cap \mathcal{L}(\Gamma) = \mathbb{C} \iff \Gamma$ is icc relatively to Λ.
- $[\mathcal{L}(\Gamma) : \mathcal{L}(\Lambda)] = [\Gamma : \Lambda] < \infty$.

We proceed to consider when $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ is C^*-irreducible. A few quick facts (the second follows from Popa's theorem):

- $C^*_\lambda(\Lambda)' \cap C^*_\lambda(\Gamma) = \mathbb{C} \iff \Gamma$ is icc rel. to Λ.
- For $|\Gamma : \Lambda| < \infty$: $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ C^*-irreduc. $\iff$ Γ icc rel. to Λ and Γ C^*-simple.
- $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ C^*-irreducible $\not\Rightarrow [\Gamma : \Lambda] < \infty$.

Theorem: Let $\Lambda \subseteq \Gamma$ be groups.

(i) $\exists\, \Gamma \curvearrowright X$ top. free bdry action st $\forall \mu \in \mathrm{Prob}(X)\ \exists\, \delta_x \in \overline{\Lambda.\mu}$ for which Γ acts freely on x,

(ii) $\tau_0 \in \overline{\{s.\varphi : s \in \Lambda\}}^{\text{weak}^*}$, for all states φ on $C^*_\lambda(\Gamma)$,

(iii) $\tau_0 \in \overline{\mathrm{conv}\{s.\varphi : s \in \Lambda\}}^{\text{weak}^*}$, for all states φ on $C^*_\lambda(\Gamma)$,

(iv) The relative Powers' averaging procedure holds: $\forall s_1, \dots, s_n \in \Gamma \setminus \{e\}$ $\forall \varepsilon > 0\ \exists t_1, \dots, t_m \in \Lambda$ st $\|\frac{1}{m}\sum_{k=1}^m \lambda(t_k s_j t_k^{-1})\| \leq \varepsilon$, for $j = 1, \dots, n$.

(v) $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ has the relative Dixmier property,

(vi) $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ is C^*-irreducible.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (vi), & (vi) $\Rightarrow$ (v) if $[\Gamma : \Lambda] < \infty$.

- (vi) $\Rightarrow$ (v) when $[\Gamma : \Lambda] < \infty$ follows from Popa's thm.
- Condition (iv) is termed $\Lambda \subseteq \Gamma$ is *plump* by Amrutam–Ursu.

Example: $C^*_\lambda(\mathbb{F}_n) \subseteq C^*_\lambda(\mathbb{F}_m)$ is C^*-irreducible when $n < m$.

(i) $\exists\, \Gamma \curvearrowright X$ top. free bdry action st $\forall \mu \in \mathrm{Prob}(X)\ \exists\, \delta_x \in \overline{\Lambda.\mu}$ for which Γ acts freely on x,

(ii) $\tau_0 \in \overline{\{s.\varphi : s \in \Lambda\}}^{\text{weak}^*}$, for all states φ on $C^*_\lambda(\Gamma)$,

(iii) $\tau_0 \in \overline{\mathrm{conv}\{s.\varphi : s \in \Lambda\}}^{\text{weak}^*}$, for all states φ on $C^*_\lambda(\Gamma)$,

(iv) The relative Powers' averaging procedure holds: $\forall s_1, \dots, s_n \in \Gamma \setminus \{e\}$ $\forall \varepsilon > 0\ \exists t_1, \dots, t_m \in \Lambda$ st $\|\frac{1}{m} \sum_{k=1}^m \lambda(t_k s_j t_k^{-1})\| \leq \varepsilon$, for $j = 1, \dots, n$.

(v) $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ has the relative Dixmier property,

(vi) $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ is C^*-irreducible.

Ursu: For Λ is **normal** in Γ: (iv) $\implies$ $\Gamma \curvearrowright \partial_F \Lambda$ is free $\implies$ (i). Hence (i)–(v) are equivalent.

Bedos–Omland: For Λ is **normal** in Γ: (i)–(vi) are equivalent, and also equiv. to Γ icc rel. to Λ, i.e., $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ is irreducible.

Bedos–Omland: $\exists$ C^*-simple groups $\Lambda \subseteq \Gamma$ with Γ icc rel. to Λ st $C^*_\lambda(\Lambda) \subseteq C^*_\lambda(\Gamma)$ not C^*-irreducible.

Outline

1 C^*-irreducible inclusions

2 von Neumann algebras

3 Inductive limits and tensor products

4 Inclusions arising from groups and dynamics

5 Bisch–Haagerup type inclusions

In their Annales Scient. ENS paper from 1996, Bisch and Haagerup considered inclusions of II_1-factors of the from $\mathcal{P}^H \subseteq \mathcal{P} \rtimes G$, where P is a type II_1-factor (e.g., $\mathcal{P} = \mathcal{R}$) and H and G are finite groups acting outerly on $\mathcal{P}$, i.e., $G, H \subseteq \mathrm{Out}(\mathcal{P})$.

Theorem [Bisch-Haagerup]:

(i) $\mathcal{R}^H \subseteq \mathcal{R} \rtimes G$ finite depth $\Leftrightarrow \langle G, H \rangle$ finite,

(ii) $\mathcal{R}^H \subseteq \mathcal{R} \rtimes G$ amenable $\Leftrightarrow \langle G, H \rangle$ amenable,

(iii) For any II_1-factor $\mathcal{P}$: $\mathcal{P}^H \subseteq \mathcal{P} \rtimes G$ irreducible $\Leftrightarrow G \cap H = \{\mathrm{id}_{\mathcal{P}}\}$.

Given actions $G \curvearrowright \mathcal{A}$ and $H \curvearrowright \mathcal{A}$ on a C^*-alg $\mathcal{A}$, with H finite. What can we say about inclusions of C^*-algs: $\mathcal{A}^H \subseteq \mathcal{A} \rtimes G$?

Theorem [Izumi]: If H is a finite group acting outerly on a unital simple C^*-alg $\mathcal{A}$, then

(i) $\mathcal{A}^H \subseteq \mathcal{D} \subseteq \mathcal{A} \implies \mathcal{D} = \mathcal{A}^L$, for some $L \subseteq H$,

(ii) $\mathcal{A}^H \subseteq \mathcal{A}$ is C^*-irreducible.

Theorem [Echterhoff-R]: Given $G \curvearrowright \mathcal{A}$ and $H \curvearrowright \mathcal{A}$ with H finite, where $\mathcal{A}$ = unital simple C^*-alg. TFAE:

(i) $\mathcal{A}^H \subseteq \mathcal{A} \rtimes_r G$ C^*-irreducible,

(ii) $(\mathcal{A}^H)' \cap (\mathcal{A} \rtimes_r G) = \mathbb{C}$,

(iii) $G \cap H = \{\mathrm{id}\}$ in $\mathrm{Out}(\mathcal{A})$.

Moreover, when H abelian and the actions of G and H commute, there is a Galois type correspondance between intermediate C^*-algs of $\mathcal{A}^H \subseteq \mathcal{A} \rtimes_r G$ and subgroups of $\widehat{H} \times G$.

The proof uses C^*-irreducibility of $\mathcal{A}^H \subseteq \mathcal{A}$ and the following:

Lemma [Echterhoff-R]: Assuming (iii) above:

$\forall x \in \mathcal{A} \rtimes_r G \quad \forall \varepsilon > 0 \quad \exists h \in (\mathcal{A}^H)^+$ with $\|h\| = 1$ st

$$\|h(x - E_{\mathcal{A}}(x))h\| < \varepsilon, \qquad \|hE_{\mathcal{A}}(x)h\| \geq \|E_{\mathcal{A}}(x)\| - \varepsilon.$$

Examples: Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\mathcal{A}_\theta$ = associated irr. rotation alg. There is a canonical outer action $\alpha \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathcal{A}_\theta)$ given by

$$\alpha_x(u) = e^{2\pi i x_{11} x_{21} \theta} u^{x_{11}} v^{x_{21}}, \qquad \alpha_x(v) = e^{2\pi i x_{12} x_{22} \theta} u^{x_{12}} v^{x_{22}},$$

for $x \in \mathrm{SL}_2(\mathbb{Z})$.

Up to conjugacy $\exists$ precisely 4 finite cyclic subgroups of $\mathrm{SL}_2(\mathbb{Z})$ isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$.

Theorem [Echterhoff, Luck, Phillips, Walters]: $(\mathcal{A}_\theta)^{\mathbb{Z}/k}$ and $\mathcal{A}_\theta \rtimes \mathbb{Z}/k$ are simple AF-algebras, for $k = 2, 3, 4, 6$ (and K-theory is computed).

The actions of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ commute, while the actions of $\mathbb{Z}_3$ and $\mathbb{Z}_4$ don't.

Corollary (Echterhoff-R): For $(F_1, F_2) = (\mathbb{Z}_2, \mathbb{Z}_3), (\mathbb{Z}_3, \mathbb{Z}_4), (\mathbb{Z}_3, \tilde{\mathbb{Z}}_3)$, the inclusions

$$(\mathcal{A}_\theta)^{F_1} \subseteq \mathcal{A}_\theta \rtimes F_2, \qquad (\mathcal{A}_\theta)^{F_2} \subseteq \mathcal{A}_\theta \rtimes F_1$$

are C^*-irreducible inclusion of AF-algebras with non-AF intermediate C^*-alg, namely $\mathcal{A}_\theta$.