On ℓ-open C^*-algebras and semiprojectivity

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- Let $A = C_0((0,1])$.Then every *-homomorphisms $\phi : C_0(0,1] \to B/I$ are liftable.

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$$d(\phi, \psi) = \sup_j \|\phi(a_j) - \psi(a_j)\|, \quad \phi, \psi \in Hom(A, B).$$

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- What conditions on A ensure that the point-norm limit of a sequence of liftable *-homomorphisms $\phi_{\boldsymbol{n}} : A \to B/I$ are liftable?

An Example (Blackadar '16)

Let A be the universal C^*-algebra generated by a sequence of projections $\{p_1, p_2, \ldots\}$

- Let $B = C([0,1]), I = C_0((0,1))$. Then $B/I \cong \mathbb{C} \oplus \mathbb{C}$.

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- ϕ_n converges point-norm to the zero homomorphism.

ℓ-open and ℓ-closed C^*-algebras

Definition (Blackadar '16)

Let A be a separable C^*-algebra and $Hom(A, B, I)$ be space of liftable *-homomorphism from A to B/I.

- A is ℓ-open (or ℓ-closed) if for every separable C^*-algebras B and closed ideal I of B, $Hom(A, B, I)$ is open (or closed) in $Hom(A, B/I)$.

Semiprojective C^*-algebra

$$\begin{tikzcd} & B/I_n \arrow[d, "\pi_n"] \\ A \arrow[ur, dashed, "\overline{\phi}"] \arrow[r, "\phi"'] & B/\overline{\bigcup_n I_n} \end{tikzcd}$$

$$I_n \lhd I_{n+1} \lhd I = \overline{\bigcup_n I_n} \lhd B$$

Some examples of semiprojective C^*-algebras

- Finite dimensional C^*-algebras,
- The universal C^*-algebras generated by finite unitaries, $C^*(\mathbb{F}_n)$,
- $\{f \in C(S^1, \mathrm{M}_n) : f(1)\,\text{is scalar}\,\}$.

Lifting close *-homomorphisms from ℓ-open C^*-algebras

Theorem 1 (O-Tikuisis)

Let A be a ℓ-open C^*-algebra generated by a finite or countable set $\mathcal{G} = \{a_1, a_2, \ldots\}$ with $\lim\limits_{n\to\infty} \|a_n\| = 0$ if $\mathcal{G}$ is infinite. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that whenever B is a separable C*-algebra, I is a closed ideal of B, ψ and ϕ are *-homomorphisms from A to B/I with $\|\phi(a_j) - \psi(a_j)\| < \delta$ for all j and such that ϕ lifts to a *-homomorphism $\overline{\phi} : A \to B$, then ψ also lifts to a *-homomorphism $\overline{\psi} : A \to B$ with $\|\overline{\psi}(a_j) - \overline{\phi}(a_j)\| < \epsilon$ for all j.

The following consequences follow using the ideas of Blackadar.

Corollary 1

Let A be a ℓ-open C^*-algebra. Then $Hom(A, B)$ is locally path-connected for any separable C^*-algebra B.

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Example

Let $X = \{(x, y) : y = sin(\frac{\pi}{x}), 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$. Then $C(X)$ is not ℓ-open.

Homotopy Lifting Theorem

Let A be a ℓ-open C^*-algebra, B a separable C^*-algebra, I a closed ideal of B, $(\phi_t)(0 \leq t \leq 1)$ a point-norm continuous path of *-homomorphism from A to B/I.

Homotopy Lifting Theorem

Let A be a ℓ-open C^*-algebra, B a separable C^*-algebra, I a closed ideal of B, $(\phi_t)(0 \leq t \leq 1)$ a point-norm continuous path of *-homomorphism from A to B/I. Suppose ϕ_0 lifts to a *-homomorphism $\overline{\phi_0} : A \to B$. Then there is a point-norm continuous path $(\overline{\phi_t})(0 \leq t \leq 1)$ of *-homomorphisms from A to B starting at $\overline{\phi_0}$ such that $\overline{\phi_t}$ is a lift of ϕ_t

Combining all the previous theorems and corollaries, we have the following characterization of ℓ-open C^*-algebra.

Theorem 2 (0-Tikuisis)

Let A be a separable C^*-algebra. Then the following are equivalent

1. A is ℓ-open
2. A satisfies the conclusion of Theorem 1
3. $Hom(A, B)$ is locally-path connected for all separable C^*-algebras B and A satisfies the conclusion of the homotopy lifting theorem.

An immediate consequence of Theorem 2 confirms a conjecture of Blackadar.

Corollary 2 (O-Tikuisis)

Let A be a ℓ-open C^*-algebra. Then A is ℓ-closed.

A C^*-algebra which is ℓ-closed but not ℓ-open (Blackadar '16)

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- Let B be a C^*-algebra, I an ideal of B, and $\phi_n : A \to B/I$ defined by

  $$\phi_n(p_k) = q_k^{(n)}$$

  be a sequence of liftable *-homomorphisms.

- Suppose ϕ_n converges point-norm to a *-homomorphism $\phi : A \to B/I$ defined by

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- Then $q_k^{(n)} \to q_k$.

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- Then $q_k^{(n)} \to q_k$.
- Semiprojectivity of $\mathbb{C}$ implies q_k lifts for each k. Hence, A is ℓ-closed.

Theorem (Sorensen-Thiel ' 12)

Let $C(X)$ be a unital, separable C^*-algebra. Then the following are equivalent

1. $C(X)$ is a semiprojective C^*-algebra
2. X is an ANR and $dim(X) \leq 1$

Theorem (O-Tikuisis)

Let X be a compact metrizable space. Then the following are equivalent

1. $C(X)$ is a semiprojective C^*-algebra
2. $C(X)$ is a ℓ-open C^*-algebra
3. X is a ANR and $dim(X) \leq 1$

Some Future Research Directions

- Investigation of whether ℓ-open C^*-algebras coincide with semiprojective C^* algebras in general.
- Characterization of ℓ-closed C^*-algebras.
- Possible application of homotopy lifting theorem of ℓ-open C^*-algebras.

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Thank you for your attention.