

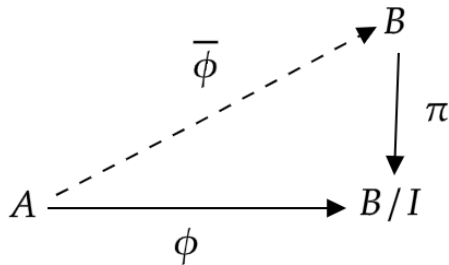
On ℓ -open C^* -algebras and semiprojectivity

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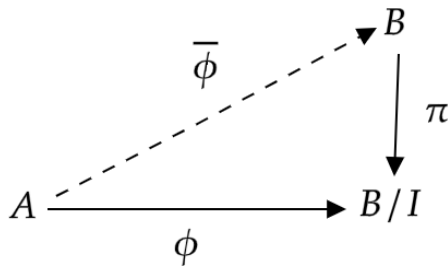
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Liftable $*$ -homomorphism

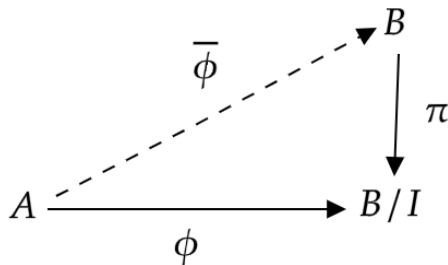


Liftable $*$ -homomorphism



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Liftable $*$ -homomorphism



- Let $A = C_0((0, 1])$. Then every $*$ -homomorphism $\phi : C_0(0, 1] \rightarrow B/I$ are liftable.

Space of $*$ -homomorphisms

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$$d(\phi, \psi) = \sup_j \|\phi(a_j) - \psi(a_j)\|, \quad \phi, \psi \in \text{Hom}(A, B).$$

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Questions

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- What conditions on A ensure that the point-norm limit of a sequence of liftable $*$ -homomorphisms $\phi_n : A \rightarrow B/I$ are liftable?

An Example (Blackadar '16)

Let A be the universal C^* -algebra generated by a sequence of projections $\{p_1, p_2, \dots\}$

- Let $B = C([0, 1])$, $I = C_0((0, 1))$. Then $B/I \cong \mathbb{C} \oplus \mathbb{C}$.

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- ϕ_n converges point-norm to the zero homomorphism.

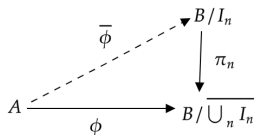
ℓ -open and ℓ -closed C^* -algebras

Definition (Blackadar '16)

Let A be a separable C^* -algebra and $\text{Hom}(A, B, I)$ be space of liftable $*$ -homomorphism from A to B/I .

- A is ℓ -open (or ℓ -closed) if for every separable C^* -algebras B and closed ideal I of B , $\text{Hom}(A, B, I)$ is open (or closed) in $\text{Hom}(A, B/I)$.

Semiprojective C^* -algebra



$$I_n \triangleleft I_{n+1} \triangleleft I = \overline{\bigcup_n I_n} \triangleleft B$$

Some examples of semiprojective C^* -algebras

- Finite dimensional C^* -algebras,
- The universal C^* -algebras generated by finite unitaries, $C^*(\mathbb{F}_n)$,
- $\{f \in C(S^1, M_n) : f(1) \text{ is scalar}\}.$

Lifting close $*$ -homomorphisms from ℓ -open C^* -algebras

Theorem 1 (O-Tikuisis)

Let A be a ℓ -open C^* -algebra generated by a finite or countable set $\mathcal{G} = \{a_1, a_2, \dots\}$ with $\lim_{n \rightarrow \infty} \|a_n\| = 0$ if \mathcal{G} is infinite. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that whenever B is a separable C^* -algebra, I is a closed ideal of B , ψ and ϕ are $*$ -homomorphisms from A to B/I with $\|\phi(a_j) - \psi(a_j)\| < \delta$ for all j and such that ϕ lifts to a $*$ -homomorphism $\bar{\phi} : A \rightarrow B$, then ψ also lifts to a $*$ -homomorphism $\bar{\psi} : A \rightarrow B$ with $\|\bar{\psi}(a_j) - \bar{\phi}(a_j)\| < \epsilon$ for all j .

The following consequences follow using the ideas of Blackadar.

Corollary 1

Let A be a ℓ -open C^* -algebra. Then $\text{Hom}(A, B)$ is locally path-connected for any separable C^* -algebra B .

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Example

Let $X = \{(x, y) : y = \sin(\frac{\pi}{x}), 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$. Then $C(X)$ is not ℓ -open.

Homotopy Lifting Theorem

Let A be a ℓ -open C^* -algebra, B a separable C^* -algebra, I a closed ideal of B , $(\phi_t)(0 \leq t \leq 1)$ a point-norm continuous path of $*$ -homomorphism from A to B/I .

Homotopy Lifting Theorem

Let A be a ℓ -open C^* -algebra, B a separable C^* -algebra, I a closed ideal of B , $(\phi_t)(0 \leq t \leq 1)$ a point-norm continuous path of $*$ -homomorphism from A to B/I . Suppose ϕ_0 lifts to a $*$ -homomorphism $\overline{\phi_0} : A \rightarrow B$. Then there is a point-norm continuous path $(\overline{\phi_t})(0 \leq t \leq 1)$ of $*$ -homomorphisms from A to B starting at $\overline{\phi_0}$ such that $\overline{\phi_t}$ is a lift of ϕ_t

Combining all the previous theorems and corollaries, we have the following characterization of ℓ -open C^* -algebra.

Theorem 2 (O-Tikuisis)

Let A be a separable C^* -algebra. Then the following are equivalent

- 1 A is ℓ -open
- 2 A satisfies the conclusion of Theorem 1
- 3 $\text{Hom}(A, B)$ is locally-path connected for all separable C^* -algebras B and A satisfies the conclusion of the homotopy lifting theorem.

An immediate consequence of Theorem 2 confirms a conjecture of Blackadar.

Corollary 2 (O-Tikuisis)

Let A be a ℓ -open C^* -algebra. Then A is ℓ -closed.

A C^* -algebra which is ℓ -closed but not ℓ -open (Blackadar '16)

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- Let B be a C^* -algebra, I an ideal of B , and $\phi_n : A \rightarrow B/I$ defined by

$$\phi_n(p_k) = q_k^{(n)}$$

be a sequence of liftable $*$ -homomorphisms.

- Suppose ϕ_n converges point-norm to a $*$ -homomorphism $\phi : A \rightarrow B/I$ defined by

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- Then $q_k^{(n)} \rightarrow q_k$.

A C^* -algebra which is ℓ -closed but not ℓ -open (Blackadar '16)

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- Then $q_k^{(n)} \rightarrow q_k$.
- Semiprojectivity of \mathbb{C} implies q_k lifts for each k . Hence, A is ℓ -closed.

Theorem (Sorensen-Thiel '12)

Let $C(X)$ be a unital, separable C^* -algebra. Then the following are equivalent

- 1 $C(X)$ is a semiprojective C^* -algebra
- 2 X is an ANR and $\dim(X) \leq 1$

Theorem (O-Tikuisis)

Let X be a compact metrizable space. Then the following are equivalent

- 1 $C(X)$ is a semiprojective C^* -algebra
- 2 $C(X)$ is a ℓ -open C^* -algebra
- 3 X is a ANR and $\dim(X) \leq 1$

Some Future Research Directions

- Investigation of whether ℓ -open C^* -algebras coincide with semiprojective C^* algebras in general.
- Characterization of ℓ -closed C^* -algebras.
- Possible application of homotopy lifting theorem of ℓ -open C^* -algebras.

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Thank you for your attention.