Convex structure of unital quantum channels, factorizability and traces on the universal free product of matrix algebras

Magdalena Musat
University of Copenhagen

Canadian Operator Algebras Symposium
University of Ottawa

May 31, 2022

Central theme: A certain class of completely positive maps (factorizable maps), introduced by C. Anantharaman-Delaroche '05.

Their study **led to** investigating the convex structure of the set of unital quantum channels, interesting applications in the analysis of QIT (e.g., settling in the negative the Asymptotic Quantum Birkhoff Conjecture) and revealed infinite dim phenomena therein, connections to/reformulations of the Connes Embedding Problem, and recently, through a new view-point, some interesting problems in operator algebras.

For $n \geq 2$, consider following sets of maps $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$:

$$\mathcal{FM}(n) \subseteq \mathrm{UCPT}(n) \begin{array}{l} \subsetneq \mathrm{CPT}(n) \\ \\ \subsetneq \mathrm{UCP}(n) \end{array}$$

all compact and convex subsets of $\mathbb{C}^{n^4}$.

Central theme: A certain class of completely positive maps (factorizable maps), introduced by C. Anantharaman-Delaroche '05.

Their study **led to** investigating the convex structure of the set of unital quantum channels, interesting applications in the analysis of QIT (e.g., settling in the negative the Asymptotic Quantum Birkhoff Conjecture) and revealed infinite dim phenomena therein, connections to/reformulations of the Connes Embedding Problem, and recently, through a new view-point, some interesting problems in operator algebras.

For $n \geq 2$, consider following sets of maps $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$:

$$\mathcal{FM}(n) \subseteq \mathrm{UCPT}(n) \begin{array}{l} \subsetneq \ \mathrm{CPT}(n) \\ \\ \subsetneq \ \mathrm{UCP}(n) \end{array}$$

all compact and convex subsets of $\mathbb{C}^{n^4}$.

▶ (Choi '73): Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear. Then

$$T \in \mathrm{CPT}(n) \iff \exists A_1, \dots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j, \ \sum_{j=1}^{d} A_j A_j^* = I_n.$$

$$T \in \mathrm{UCP}(n) \iff \exists A_1, \dots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j, \ \sum_{j=1}^{d} A_j^* A_j = I_n.$$

When $\{A_1, \dots, A_d\}$ lin independent, d is called the Choi-rank of T.

Thm (Choi '75): $T \in \partial_e(\mathrm{CPT}(n)) \iff \{A_i A_j^*\}_{i,j=1}^{d}$ lin independent.
Respectively, $T \in \partial_e(\mathrm{UCP}(n)) \iff \{A_i^* A_j\}_{i,j=1}^{d}$ lin independent.

Thm (Landau-Streater '93): $T \in \partial_e(\mathrm{UCPT}(n)) \iff \{A_i^* A_j \oplus A_j A_i^*\}_{i,j}$ lin independent.

Hence $\partial_e(\mathrm{UCPT}(n)) \supseteq (\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n)$.

Question: Are these sets equal?

- (Choi '73): Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear. Then

$$T \in \mathrm{CPT}(n) \iff \exists A_1, \ldots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j,\ \sum_{j=1}^{d} A_j A_j^* = I_n.$$

$$T \in \mathrm{UCP}(n) \iff \exists A_1, \ldots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j,\ \sum_{j=1}^{d} A_j^* A_j = I_n.$$

When $\{A_1, \ldots, A_d\}$ lin independent, d is called the Choi-rank of T.

Thm (Choi '75): $T \in \partial_e(\mathrm{CPT}(n)) \iff \{A_i A_j^*\}_{i,j=1}^{d}$ lin independent.
Respectively, $T \in \partial_e(\mathrm{UCP}(n)) \iff \{A_i^* A_j\}_{i,j=1}^{d}$ lin independent.

Thm (Landau-Streater '93): $T \in \partial_e(\mathrm{UCPT}(n)) \iff \{A_i^* A_j \oplus A_j A_i^*\}_{i,j}$ lin independent.

Hence $\partial_e(\mathrm{UCPT}(n)) \supseteq (\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n)$.

Question: Are these sets equal?

▶ (Choi '73): Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear. Then

$$T \in \mathrm{CPT}(n) \iff \exists A_1, \ldots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j, \ \sum_{j=1}^{d} A_j A_j^* = I_n.$$

$$T \in \mathrm{UCP}(n) \iff \exists A_1, \ldots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j, \ \sum_{j=1}^{d} A_j^* A_j = I_n.$$

When $\{A_1, \ldots, A_d\}$ lin independent, d is called the Choi-rank of T.

Thm (Choi '75): $T \in \partial_e(\mathrm{CPT}(n)) \iff \{A_i A_j^*\}_{i,j=1}^d$ lin independent.
Respectively, $T \in \partial_e(\mathrm{UCP}(n)) \iff \{A_i^* A_j\}_{i,j=1}^d$ lin independent.

Thm (Landau-Streater '93): $T \in \partial_e(\mathrm{UCPT}(n)) \iff \{A_i^* A_j \oplus A_j A_i^*\}_{i,j}$ lin independent.

Hence $\partial_e(\mathrm{UCPT}(n)) \supseteq (\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n)$.

Question: Are these sets equal?

- (Choi '73): Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear. Then

$$T \in \mathrm{CPT}(n) \iff \exists A_1, \dots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j,\ \sum_{j=1}^{d} A_j A_j^* = I_n.$$

$$T \in \mathrm{UCP}(n) \iff \exists A_1, \dots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j,\ \sum_{j=1}^{d} A_j^* A_j = I_n.$$

When $\{A_1, \dots, A_d\}$ lin independent, d is called the Choi-rank of T.

Thm (Choi '75): $T \in \partial_e(\mathrm{CPT}(n)) \iff \{A_i A_j^*\}_{i,j=1}^{d}$ lin independent.
Respectively, $T \in \partial_e(\mathrm{UCP}(n)) \iff \{A_i^* A_j\}_{i,j=1}^{d}$ lin independent.

Thm (Landau-Streater '93): $T \in \partial_e(\mathrm{UCPT}(n)) \iff \{A_i^* A_j \oplus A_j A_i^*\}_{i,j}$ lin independent.

Hence $\partial_e(\mathrm{UCPT}(n)) \supseteq \big(\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))\big) \cap \mathrm{UCPT}(n)$.

Question: Are these sets equal?

▶ (Choi '73): Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear. Then

$$T \in \mathrm{CPT}(n) \iff \exists A_1, \ldots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j, \ \sum_{j=1}^{d} A_j A_j^* = I_n.$$

$$T \in \mathrm{UCP}(n) \iff \exists A_1, \ldots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j, \ \sum_{j=1}^{d} A_j^* A_j = I_n.$$

When $\{A_1, \ldots, A_d\}$ lin independent, d is called the Choi-rank of T.

Thm (Choi '75): $T \in \partial_e(\mathrm{CPT}(n)) \iff \{A_i A_j^*\}_{i,j=1}^{d}$ lin independent. Respectively, $T \in \partial_e(\mathrm{UCP}(n)) \iff \{A_i^* A_j\}_{i,j=1}^{d}$ lin independent.

Thm (Landau-Streater '93): $T \in \partial_e(\mathrm{UCPT}(n)) \iff \{A_i^* A_j \oplus A_j A_i^*\}_{i,j}$ lin independent.

Hence $\partial_e(\mathrm{UCPT}(n)) \supseteq \big(\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))\big) \cap \mathrm{UCPT}(n)$.

Question: Are these sets equal?

Definition (Anantharaman-Delaroche '05): A unital quantum channel $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called *factorizable* if $\exists$ vN alg (N, ψ) with n.f. tracial state and unital $*$-homs $\alpha, \beta\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$: $T = \beta^* \circ \alpha$.

$$\begin{tikzcd}
M_n(\mathbb{C}) \arrow[rr, "T"] \arrow[dr, "\alpha"'] & & M_n(\mathbb{C}) \arrow[dl, "\beta"'] \\
& M_n(\mathbb{C}) \otimes N \arrow[ur, bend right, "\beta^* = \beta^{-1} \circ \mathbb{E}_{\beta(M_n(\mathbb{C}))}"'] &
\end{tikzcd}$$

▶ α, β are injective (thus embeddings) and trace-preserving. Since unital embeddings of $M_n(\mathbb{C})$ into a vN alg are unitarily equiv, can take

$$\beta(x) = x \otimes 1_N, \quad \alpha(x) = u^*\beta(x)u, \quad x \in M_n(\mathbb{C}),$$

for some $u \in M_n(\mathbb{C}) \otimes N$ unitary; N can be taken II_1-vN alg (even factor).

Theorem (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel **iff** $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

Definition (Anantharaman-Delaroche '05): A unital quantum channel $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called *factorizable* if $\exists$ vN alg (N, ψ) with n.f. tracial state and unital $*$-homs $\alpha, \beta\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$: $T = \beta^* \circ \alpha$.

$$\begin{tikzcd}
M_n(\mathbb{C}) \arrow[rr, "T"] \arrow[dr, "\alpha"'] & & M_n(\mathbb{C}) \arrow[dl, "\beta"'] \\
& M_n(\mathbb{C}) \otimes N \arrow[ur, bend right, "\beta^* = \beta^{-1} \circ \mathbb{E}_{\beta(M_n(\mathbb{C}))}"'] &
\end{tikzcd}$$

▶ α, β are injective (thus embeddings) and trace-preserving. Since unital embeddings of $M_n(\mathbb{C})$ into a vN alg are unitarily equiv, can take

$$\beta(x) = x \otimes 1_N, \quad \alpha(x) = u^* \beta(x) u, \quad x \in M_n(\mathbb{C}),$$

for some $u \in M_n(\mathbb{C}) \otimes N$ unitary; N can be taken II_1-vN alg (even factor).

Theorem (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel **iff** $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

Def/Thm (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel **iff** $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

- (R. Werner): Factorizable channels are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.

- Automorphisms of $M_n(\mathbb{C})$ are factorizable.

Let $\mathcal{FM}(n)$ denote all factorizable quantum channels on $M_n(\mathbb{C})$, $n \geq 2$. Then $\mathcal{FM}(n)$ is convex and closed.

Proposition (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, with *canonical form*

$$Tx = \sum_{i=1}^{d} A_j^* x A_j, \quad x \in M_n(\mathbb{C}).$$

If $d := \text{Choi-rank}(T) \geq 2$ and $\{A_i^* A_j\}_{1 \leq i,j}^{d}$ lin indep, then $T \notin \mathcal{FM}(n)$.

Def/Thm (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel **iff** $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

- (R. Werner): Factorizable channels are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.
- Automorphisms of $M_n(\mathbb{C})$ are factorizable.

Let $\mathcal{FM}(n)$ denote all factorizable quantum channels on $M_n(\mathbb{C})$, $n \geq 2$. Then $\mathcal{FM}(n)$ is convex and closed.

Proposition (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, with *canonical form*

$$Tx = \sum_{i=1}^{d} A_j^* x A_j, \quad x \in M_n(\mathbb{C}).$$

If $d := \text{Choi-rank}(T) \geq 2$ and $\{A_i^* A_j\}_{1 \leq i,j}^{d}$ lin indep, then $T \notin \mathcal{FM}(n)$.

Def/Thm (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel **iff** $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

- (R. Werner): Factorizable channels are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.
- Automorphisms of $M_n(\mathbb{C})$ are factorizable.

Let $\mathcal{FM}(n)$ denote all factorizable quantum channels on $M_n(\mathbb{C})$, $n \geq 2$. Then $\mathcal{FM}(n)$ is convex and closed.

Proposition (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, with *canonical form*

$$Tx = \sum_{i=1}^{d} A_j^* x A_j, \quad x \in M_n(\mathbb{C}).$$

If $d := \text{Choi-rank}(T) \geq 2$ and $\{A_i^* A_j\}_{1 \leq i,j}^{d}$ lin indep, then $T \notin \mathcal{FM}(n)$.

Def/Thm (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel **iff** $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

- (R. Werner): Factorizable channels are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.

- Automorphisms of $M_n(\mathbb{C})$ are factorizable.

Let $\mathcal{FM}(n)$ denote all factorizable quantum channels on $M_n(\mathbb{C})$, $n \geq 2$. Then $\mathcal{FM}(n)$ is convex and closed.

Proposition (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, with *canonical form*

$$Tx = \sum_{i=1}^{d} A_j^* x A_j, \quad x \in M_n(\mathbb{C}).$$

If $d := \text{Choi-rank}(T) \geq 2$ and $\{A_i^* A_j\}_{1 \leq i,j}^{d}$ lin indep, then $T \notin \mathcal{FM}(n)$.

- (Kümmerer '83): $\mathrm{UCPT}(2) = \mathrm{conv}(\mathrm{Aut}(M_2(\mathbb{C})))$. However,

$$\mathrm{UCPT}(n) \supsetneq \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))), \quad n \geq 3,$$

cf. (Kümmerer '86, Landau-Streater '93, Kümmerer-Maasen '87).

- **Asymptotic Quantum Birkhoff Conj** (Smolin-Verstraete-Winter '05): Any $T \in \mathrm{UCPT}(n)$, $n \geq 3$, satisfies

$$\lim_{k\to\infty} d_{\mathrm{cb}}\Big(\bigotimes_{i=1}^{k} T\,, \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\Big) = 0\,.$$

Theorem (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, $n \geq 3$. For $k \geq 1$,

$$d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T\,, \mathcal{F}\left(\bigotimes_{i=1}^{k} M_n(\mathbb{C})\right)\right) \geq d_{\mathrm{cb}}(T,\ \mathcal{F}\mathcal{M}(n))\,.$$

- If $T \notin \mathcal{F}\mathcal{M}(n))$, then $d_{\mathrm{cb}}(T\,, \mathcal{F}\mathcal{M}(n)) > 0$.

As $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subset \mathcal{F}\mathcal{M}(n)$, any non-factoriz channel fails **AQBC**.

▶ (Kümmerer '83): $\mathrm{UCPT}(2) = \mathrm{conv}(\mathrm{Aut}(M_2(\mathbb{C})))$. However,

$$\mathrm{UCPT}(n) \supsetneq \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) , \quad n \geq 3 ,$$

cf. (Kümmerer '86, Landau-Streater '93, Kümmerer-Maasen '87).

▶ **Asymptotic Quantum Birkhoff Conj** (Smolin-Verstraete-Winter '05): Any $T \in \mathrm{UCPT}(n)$, $n \geq 3$, satisfies

$$\lim_{k \to \infty} d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T , \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C}))) \right) = 0 .$$

Theorem (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, $n \geq 3$. For $k \geq 1$,

$$d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T , \mathcal{F}\left(\bigotimes_{i=1}^{k} M_n(\mathbb{C}) \right) \right) \geq d_{\mathrm{cb}}(T,\ \mathcal{FM}(n)) .$$

▶ If $T \notin \mathcal{FM}(n)$, then $d_{\mathrm{cb}}(T , \mathcal{FM}(n)) > 0$.
As $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subset \mathcal{FM}(n)$, any non-factoriz channel fails **AQBC**.

▶ (Kümmerer '83): $\mathrm{UCPT}(2) = \mathrm{conv}(\mathrm{Aut}(M_2(\mathbb{C})))$. However,

$$\mathrm{UCPT}(n) \supsetneq \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))), \quad n \geq 3,$$

cf. (Kümmerer '86, Landau-Streater '93, Kümmerer-Maasen '87).

▶ **Asymptotic Quantum Birkhoff Conj** (Smolin-Verstraete-Winter '05): Any $T \in \mathrm{UCPT}(n)$, $n \geq 3$, satisfies

$$\lim_{k\to\infty} d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T, \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\right) = 0.$$

Theorem (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, $n \geq 3$. For $k \geq 1$,

$$d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T, \mathcal{F}\left(\bigotimes_{i=1}^{k} M_n(\mathbb{C})\right)\right) \geq d_{\mathrm{cb}}(T, \mathcal{FM}(n)).$$

▶ If $T \notin \mathcal{FM}(n)$, then $d_{\mathrm{cb}}(T, \mathcal{FM}(n)) > 0$.

As $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subset \mathcal{FM}(n)$, any non-factoriz channel fails **AQBC**.

▶ (Kümmerer '83): $\mathrm{UCPT}(2) = \mathrm{conv}(\mathrm{Aut}(M_2(\mathbb{C})))$. However,

$$\mathrm{UCPT}(n) \supsetneq \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))\,, \quad n \geq 3\,,$$

cf. (Kümmerer '86, Landau-Streater '93, Kümmerer-Maasen '87).

▶ **Asymptotic Quantum Birkhoff Conj** (Smolin-Verstraete-Winter '05): Any $T \in \mathrm{UCPT}(n)$, $n \geq 3$, satisfies

$$\lim_{k\to\infty} d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T\,, \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\right) = 0\,.$$

Theorem (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, $n \geq 3$. For $k \geq 1$,

$$d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T\,, \mathcal{F}\left(\bigotimes_{i=1}^{k} M_n(\mathbb{C})\right)\right) \geq d_{\mathrm{cb}}(T,\ \mathcal{F}\mathcal{M}(n))\,.$$

▶ If $T \notin \mathcal{F}\mathcal{M}(n))$, then $d_{\mathrm{cb}}(T\,, \mathcal{F}\mathcal{M}(n)) > 0$.
As $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subset \mathcal{F}\mathcal{M}(n)$, any non-factoriz channel fails **AQBC**.

- For $T \in \mathrm{UCPT}(n)$, Choi-rank$(T) = 1$ **iff** $T \in \mathrm{Aut}(M_n(\mathbb{C}))$.

Set $\partial_e^*(\mathrm{CPT}(n)) = \partial_e(\mathrm{CPT}(n)) \setminus \mathrm{Aut}(M_n(\mathbb{C}))$, and similarly $\partial_e^*(\mathrm{UCP}(n))$.

Cor: $T \in (\partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n) \Rightarrow T \notin \mathcal{FM}(n)$.

Remark: Not easy to characterize non-factorizability in terms of the convex structure of $\mathrm{UCPT}(n)$:

- $\partial_e(\mathrm{UCPT}(n)) \setminus ((\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n))$ can contain both factorizable and non-factorizable maps.

- $T \in \partial_e^*(\mathrm{UCPT}(n))$, Choi-rank $> n \Rightarrow T \notin \partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))$.
- (Ohno '09): $\exists T \in \partial_e^*(\mathrm{UCPT}(3))$, Choi-rank 4; (H-M-R): $T \notin \mathcal{FM}(3)$.
- (H-M-R '21): Explicit family $T_t \in \partial_e^*(\mathrm{UCPT}(3)) \cap \mathcal{FM}(3)$, Choi-rank 4.

- For $T \in \mathrm{UCPT}(n)$, Choi-rank$(T) = 1$ **iff** $T \in \mathrm{Aut}(M_n(\mathbb{C}))$.

Set $\partial_e^*(\mathrm{CPT}(n)) = \partial_e(\mathrm{CPT}(n)) \setminus \mathrm{Aut}(M_n(\mathbb{C}))$, and similarly $\partial_e^*(\mathrm{UCP}(n))$.

Cor: $T \in (\partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n) \Rightarrow T \notin \mathcal{FM}(n)$.

Remark: Not easy to characterize non-factorizability in terms of the convex structure of $\mathrm{UCPT}(n)$:

- $\partial_e(\mathrm{UCPT}(n)) \setminus ((\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n))$ can contain both factorizable and non-factorizable maps.

- $T \in \partial_e^*(\mathrm{UCPT}(n))$, Choi-rank $> n \Rightarrow T \notin \partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))$.
- (Ohno '09): $\exists T \in \partial_e^*(\mathrm{UCPT}(3))$, Choi-rank 4; (H-M-R): $T \notin \mathcal{FM}(3)$.
- (H-M-R '21): Explicit family $T_t \in \partial_e^*(\mathrm{UCPT}(3)) \cap \mathcal{FM}(3)$, Choi-rank 4.

▶ For $T \in \mathrm{UCPT}(n)$, Choi-rank$(T) = 1$ **iff** $T \in \mathrm{Aut}(M_n(\mathbb{C}))$.
Set $\partial_e^*(\mathrm{CPT}(n)) = \partial_e(\mathrm{CPT}(n)) \setminus \mathrm{Aut}(M_n(\mathbb{C}))$, and similarly $\partial_e^*(\mathrm{UCP}(n))$.

Cor: $T \in (\partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n) \Rightarrow T \notin \mathcal{FM}(n)$.

Remark: Not easy to characterize non-factorizability in terms of the convex structure of $\mathrm{UCPT}(n)$:

- $\partial_e(\mathrm{UCPT}(n)) \setminus \left((\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))\right) \cap \mathrm{UCPT}(n))$ can contain both factorizable and non-factorizable maps.

▶ $T \in \partial_e^*(\mathrm{UCPT}(n))$, Choi-rank $> n \Rightarrow T \notin \partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))$.

▶ (Ohno '09): $\exists T \in \partial_e^*(\mathrm{UCPT}(3))$, Choi-rank 4; (H-M-R): $T \notin \mathcal{FM}(3)$.

▶ (H-M-R '21): Explicit family $T_t \in \partial_e^*(\mathrm{UCPT}(3)) \cap \mathcal{FM}(3)$, Choi-rank 4.

▶ For $T \in \mathrm{UCPT}(n)$, Choi-rank$(T) = 1$ **iff** $T \in \mathrm{Aut}(M_n(\mathbb{C}))$.
Set $\partial_e^*(\mathrm{CPT}(n)) = \partial_e(\mathrm{CPT}(n)) \setminus \mathrm{Aut}(M_n(\mathbb{C}))$, and similarly $\partial_e^*(\mathrm{UCP}(n))$.

Cor: $T \in (\partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n) \Rightarrow T \notin \mathcal{FM}(n)$.

Remark: Not easy to characterize non-factorizability in terms of the convex structure of $\mathrm{UCPT}(n)$:

- $\partial_e(\mathrm{UCPT}(n)) \setminus \big((\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))\big) \cap \mathrm{UCPT}(n)\big)$ can contain both factorizable and non-factorizable maps.

▶ $T \in \partial_e^*(\mathrm{UCPT}(n))$, Choi-rank $> n \Rightarrow T \notin \partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))$.

▶ (Ohno '09): $\exists T \in \partial_e^*(\mathrm{UCPT}(3))$, Choi-rank 4; (H-M-R): $T \notin \mathcal{FM}(3)$.

▶ (H-M-R '21): Explicit family $T_t \in \partial_e^*(\mathrm{UCPT}(3)) \cap \mathcal{FM}(3)$, Choi-rank 4.

- For $T \in \mathrm{UCPT}(n)$, Choi-rank$(T) = 1$ **iff** $T \in \mathrm{Aut}(M_n(\mathbb{C}))$.

Set $\partial_e^*(\mathrm{CPT}(n)) = \partial_e(\mathrm{CPT}(n)) \setminus \mathrm{Aut}(M_n(\mathbb{C}))$, and similarly $\partial_e^*(\mathrm{UCP}(n))$.

Cor: $T \in (\partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n) \Rightarrow T \notin \mathcal{FM}(n)$.

Remark: Not easy to characterize non-factorizability in terms of the convex structure of $\mathrm{UCPT}(n)$:

- $\partial_e(\mathrm{UCPT}(n)) \setminus \big((\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n)\big)$ can contain both factorizable and non-factorizable maps.

- $T \in \partial_e^*(\mathrm{UCPT}(n))$, Choi-rank $> n \Rightarrow T \notin \partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))$.
- (Ohno '09): $\exists T \in \partial_e^*(\mathrm{UCPT}(3))$, Choi-rank 4; (H-M-R): $T \notin \mathcal{FM}(3)$.
- (H-M-R '21): Explicit family $T_t \in \partial_e^*(\mathrm{UCPT}(3)) \cap \mathcal{FM}(3)$, Choi-rank 4.

A class of UCPT(n) maps constructed in (Haagerup-M-Ruskai '21):

Given $n \geq 3$, $V_1, \ldots, V_n \in \mathcal{U}(n-1)$ and $t \in [-1, 1]$, $t \neq -1/(n-1)$, set

$$A_m = \frac{1}{\sqrt{n-1-t^2}} S^{-m+1} \begin{pmatrix} t & 0 \\ 0 & V_m \end{pmatrix} S^{m-1}, \quad 1 \leq m \leq n.$$

Here S is the canonical shift on $\mathbb{C}^n$.

- Can verify $\sum_{m=1}^n A_m^* A_m = I_n = \sum_{m=1}^n A_m A_m^*$. Thus, if

$$Tx = \sum_{m=1}^n A_m^* x A_m, \quad x \in M_n(\mathbb{C}),$$

then $T \in \mathrm{UCPT}(n)$. Moreover, the Choi-rank of T is n.

A class of UCPT(n) maps constructed in (Haagerup-M-Ruskai '21):

Given $n \geq 3$, $V_1, \ldots, V_n \in \mathcal{U}(n-1)$ and $t \in [-1, 1]$, $t \neq -1/(n-1)$, set

$$A_m = \frac{1}{\sqrt{n-1-t^2}} S^{-m+1} \begin{pmatrix} t & 0 \\ 0 & V_m \end{pmatrix} S^{m-1}, \quad 1 \leq m \leq n.$$

Here S is the canonical shift on $\mathbb{C}^n$.

- Can verify $\sum_{m=1}^n A_m^* A_m = I_n = \sum_{m=1}^n A_m A_m^*$. Thus, if

$$Tx = \sum_{m=1}^{n} A_m^* x A_m, \quad x \in M_n(\mathbb{C}),$$

then $T \in \mathrm{UCPT}(n)$. Moreover, the Choi-rank of T is n.

Theorem A (H-M-R '21): For $n \geq 3$ and $t \in (-1, 1)$, $t \neq -1/(n-1)$, there exists $W = W^* \in \mathcal{U}(n-1)$ such that if $V_1 = \ldots = V_n = W$ and $\{A_m\}_{m=1}^n$ are as above, then $\{A_i^* A_j\}_{i,j=1}^n$ linearly independent. Hence

$$T \in \partial_e^*(UCP(n)) \cap \partial_e^*(CPT(n)),$$

thus T is non-factorizable.

Proof: Lots of linear algebra.

Theorem B (H-M-R '21): For $n \geq 3$ and $t \in (-1, 1)$, $t \neq -1/(n-1)$, the set of n-tuples $(V_1, \ldots, V_n) \in \mathcal{U}(n-1)^n$ such that $\{A_i^* A_j\}_{i,j=1}^n$ is linearly indep has co-measure 0 w.r.t. Haar measure. Hence almost all quantum channels T arising in this way are non-factorizable.

Proof: Theorem A + Algebraic Geometry.

Theorem A (H-M-R '21): For $n \geq 3$ and $t \in (-1, 1)$, $t \neq -1/(n-1)$, there exists $W = W^* \in \mathcal{U}(n-1)$ such that if $V_1 = \ldots = V_n = W$ and $\{A_m\}_{m=1}^n$ are as above, then $\{A_i^* A_j\}_{i,j=1}^n$ linearly independent. Hence

$$T \in \partial_e^*(UCP(n)) \cap \partial_e^*(CPT(n)),$$

thus T is non-factorizable.

Proof: Lots of linear algebra.

Theorem B (H-M-R '21): For $n \geq 3$ and $t \in (-1, 1)$, $t \neq -1/(n-1)$, the set of n-tuples $(V_1, \ldots, V_n) \in \mathcal{U}(n-1)^n$ such that $\{A_i^* A_j\}_{i,j=1}^n$ is linearly indep has co-measure 0 w.r.t. Haar measure. Hence almost all quantum channels T arising in this way are non-factorizable.

Proof: Theorem A + Algebraic Geometry.

Infinite dimensional phenomena in QIT:

Question: Do we need (inf dim) vN algs to describe factorizable channels?

▶ For a factorizable channel, *minimal* ancilla (and its size) **not** unique.
E.g., consider the completely depolarizing channel S_n, $n \geq 2$

$$S_n(x) := \mathrm{tr}_n(x)1_n = \int_{\mathcal{U}(n)} u^* x u \, d\mu(u), \quad x \in \mathbb{M}_n(\mathbb{C}).$$

It's factorizable, and possible ancillas are: $\mathbb{C}^{n^2}$, $M_n(\mathbb{C})$, but also (a corner of) the reduced free product von Neumann alg $(M_n(\mathbb{C}), \mathrm{tr}_n) * (M_n(\mathbb{C}), \mathrm{tr}_n)$.

Let $\mathcal{FM}_{\mathrm{fin}}(n)$ = factoriz channels on $M_n(\mathbb{C})$ admitting a finite dim ancilla.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\mathrm{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

Infinite dimensional phenomena in QIT:

Question: Do we need (inf dim) vN algs to describe factorizable channels?

▶ For a factorizable channel, *minimal* ancilla (and its size) **not** unique.
E.g., consider the completely depolarizing channel S_n, $n \geq 2$

$$S_n(x) := \mathrm{tr}_n(x)1_n = \int_{\mathcal{U}(n)} u^* x u \, d\mu(u), \quad x \in \mathbb{M}_n(\mathbb{C}).$$

It's factorizable, and possible ancillas are: $\mathbb{C}^{n^2}$, $M_n(\mathbb{C})$, but also (a corner of) the reduced free product von Neumann alg $(M_n(\mathbb{C}), \mathrm{tr}_n) * (M_n(\mathbb{C}), \mathrm{tr}_n)$.

Let $\mathcal{FM}_{\text{fin}}(n)$ = factoriz channels on $M_n(\mathbb{C})$ admitting a finite dim ancilla.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\text{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\text{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

Proof: Consider following sets of $n \times n$ complex matrices, $n \geq 2$:

$$\begin{aligned} \mathcal{G}_{\text{fin}}(n) &= \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite dimens C}^*\text{-alg } (\mathcal{A}, \tau) \Big\}, \\ &\cap\!| \\ \mathcal{G}(n) &= \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite vN alg } (M, \tau) \Big\}. \end{aligned}$$

- (Rørdam-M): $\mathcal{G}_{\text{fin}}(n)$ is convex $\forall n \geq 2$, but not closed whenever $n \geq 11$.

Note: If $B \in M_n(\mathbb{C})$ *correlation matrix* (pos definite, diag entries equal 1), the Schur multiplier $T_B \in UCPT(n)$. (Ricard '08): If B has real entries, then T_B is factorizable.

- (Haagerup-M '11): A Schur multiplier T_B is factorizable **iff** $B \in \mathcal{G}(n)$. Furthermore, $T_B \in \mathcal{FM}_{\text{fin}}(n)$ **iff** $B \in \mathcal{G}_{\text{fin}}(n)$.

The map $B \mapsto T_B$ is an affine homeo, hence theorem follows.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\text{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

Proof: Consider following sets of $n \times n$ complex matrices, $n \geq 2$:

$$\begin{aligned} \mathcal{G}_{\text{fin}}(n) &= \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \dots, u_n \text{ unitaries in arbitrary finite dimens C}^*\text{-alg } (\mathcal{A}, \tau) \Big\}, \\ &\cap\!\mid \\ \mathcal{G}(n) &= \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \dots, u_n \text{ unitaries in arbitrary finite vN alg } (M, \tau) \Big\}. \end{aligned}$$

- (Rørdam-M): $\mathcal{G}_{\text{fin}}(n)$ is convex $\forall n \geq 2$, but not closed whenever $n \geq 11$.

Note: If $B \in M_n(\mathbb{C})$ *correlation matrix* (pos definite, diag entries equal 1), the Schur multiplier $T_B \in UCPT(n)$. (Ricard '08): If B has real entries, then T_B is factorizable.

- (Haagerup-M '11): A Schur multiplier T_B is factorizable **iff** $B \in \mathcal{G}(n)$. Furthermore, $T_B \in \mathcal{FM}_{\text{fin}}(n)$ **iff** $B \in \mathcal{G}_{\text{fin}}(n)$.

The map $B \mapsto T_B$ is an affine homeo, hence theorem follows.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\text{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

Proof: Consider following sets of $n \times n$ complex matrices, $n \geq 2$:

$$\begin{array}{rcl} \mathcal{G}_{\text{fin}}(n) & = & \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite dimens C}^*\text{-alg } (\mathcal{A}, \tau) \Big\}, \\ \cap\!| & & \\ \mathcal{G}(n) & = & \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite vN alg } (M, \tau) \Big\}. \end{array}$$

- (Rørdam-M): $\mathcal{G}_{\text{fin}}(n)$ is convex $\forall n \geq 2$, but not closed whenever $n \geq 11$.

Note: If $B \in M_n(\mathbb{C})$ *correlation matrix* (pos definite, diag entries equal 1), the Schur multiplier $T_B \in UCPT(n)$. (Ricard '08): If B has real entries, then T_B is factorizable.

- (Haagerup-M '11): A Schur multiplier T_B is factorizable **iff** $B \in \mathcal{G}(n)$. Furthermore, $T_B \in \mathcal{FM}_{\text{fin}}(n)$ **iff** $B \in \mathcal{G}_{\text{fin}}(n)$.

The map $B \mapsto T_B$ is an affine homeo, hence theorem follows.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\text{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

Proof: Consider following sets of $n \times n$ complex matrices, $n \geq 2$:

$$\begin{aligned} \mathcal{G}_{\text{fin}}(n) &= \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite dimens C}^*\text{-alg } (\mathcal{A}, \tau) \Big\}, \\ &\cap| \\ \mathcal{G}(n) &= \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite vN alg } (M, \tau) \Big\}. \end{aligned}$$

- (Rørdam-M): $\mathcal{G}_{\text{fin}}(n)$ is convex $\forall n \geq 2$, but not closed whenever $n \geq 11$.

Note: If $B \in M_n(\mathbb{C})$ *correlation matrix* (pos definite, diag entries equal 1), the Schur multiplier $T_B \in UCPT(n)$. (Ricard '08): If B has real entries, then T_B is factorizable.

- (Haagerup-M '11): A Schur multiplier T_B is factorizable **iff** $B \in \mathcal{G}(n)$. Furthermore, $T_B \in \mathcal{FM}_{\text{fin}}(n)$ **iff** $B \in \mathcal{G}_{\text{fin}}(n)$.

The map $B \mapsto T_B$ is an affine homeo, hence theorem follows.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\text{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

Proof: Consider following sets of $n \times n$ complex matrices, $n \geq 2$:

$$\begin{aligned} \mathcal{G}_{\text{fin}}(n) &= \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \dots, u_n \text{ unitaries in arbitrary finite dimens C}^*\text{-alg } (\mathcal{A}, \tau) \Big\}, \\ &\cap\!| \\ \mathcal{G}(n) &= \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \dots, u_n \text{ unitaries in arbitrary finite vN alg } (M, \tau) \Big\}. \end{aligned}$$

- (Rørdam-M): $\mathcal{G}_{\text{fin}}(n)$ is convex $\forall n \geq 2$, but not closed whenever $n \geq 11$.

Note: If $B \in M_n(\mathbb{C})$ *correlation matrix* (pos definite, diag entries equal 1), the Schur multiplier $T_B \in UCPT(n)$. (Ricard '08): If B has real entries, then T_B is factorizable.

- (Haagerup-M '11): A Schur multiplier T_B is factorizable **iff** $B \in \mathcal{G}(n)$. Furthermore, $T_B \in \mathcal{FM}_{\text{fin}}(n)$ **iff** $B \in \mathcal{G}_{\text{fin}}(n)$.

The map $B \mapsto T_B$ is an affine homeo, hence theorem follows. $\square$

(Rørdam-M '20): A new view-point on factorizable channels, leading to further connections (and interesting open problems in C*-algebras):

- $\mathcal{FM}(n)$ is *parametrized by* simplex of tracial states $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

More precisely, if $\tau \in T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, let

$$C_\tau(i,j;k,\ell) = n\tau\big(\iota_2(e_{k\ell})^* \iota_1(e_{ij})\big), \qquad 1 \le i,j,k,\ell \le n,$$

where $\iota_1, \iota_2 \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) * M_n(\mathbb{C})$ are the canonical inclusions. Then $C_\tau \in M_{n^2}(\mathbb{C})$ is positive, hence it is the Choi matrix of some c.p. lin map $T_\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, which turns out to be a factoriz quantum channel!

In fact, the map $\Phi \colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto \Phi(\tau) := T_\tau$ is an affine continuous surjection, satisfying, moreover,

$$\Phi(T_{\mathrm{fin}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\mathrm{fin}}(n),$$

where T_{fin} = tracial states that factor through finite dim. C^*-alg.

(Rørdam-M '20): A new view-point on factorizable channels, leading to further connections (and interesting open problems in C*-algebras):

- $\mathcal{FM}(n)$ is *parametrized by* simplex of tracial states $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

More precisely, if $\tau \in T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, let

$$C_\tau(i,j;k,\ell) = n\tau\big(\iota_2(e_{k\ell})^* \, \iota_1(e_{ij})\big), \qquad 1 \le i,j,k,\ell \le n,$$

where $\iota_1, \iota_2 \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) * M_n(\mathbb{C})$ are the canonical inclusions. Then $C_\tau \in M_{n^2}(\mathbb{C})$ is positive, hence it is the Choi matrix of some c.p. lin map $T_\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, which turns out to be a factoriz quantum channel!

In fact, the map $\Phi \colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto \Phi(\tau) := T_\tau$ is an affine continuous surjection, satisfying, moreover,

$$\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n),$$

where $T_{\rm fin}$ = tracial states that factor through finite dim. C^*-alg.

(Rørdam-M '20): A new view-point on factorizable channels, leading to further connections (and interesting open problems in C*-algebras):

- $\mathcal{FM}(n)$ is *parametrized by* simplex of tracial states $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

More precisely, if $\tau \in T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, let

$$C_\tau(i,j;k,\ell) = n\tau\big(\iota_2(e_{k\ell})^* \, \iota_1(e_{ij})\big), \qquad 1 \le i,j,k,\ell \le n,$$

where $\iota_1, \iota_2 \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) * M_n(\mathbb{C})$ are the canonical inclusions. Then $C_\tau \in M_{n^2}(\mathbb{C})$ is positive, hence it is the Choi matrix of some c.p. lin map $T_\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, which turns out to be a factoriz quantum channel!

In fact, the map $\Phi\colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto \Phi(\tau) := T_\tau$ is an affine continuous surjection, satisfying, moreover,

$$\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n),$$

where $T_{\rm fin}$ = tracial states that factor through finite dim. C^*-alg.

The affine cont surj $\Phi\colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto T_\tau$, satisfies

- $\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n)$,
- $\Phi(\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}) = \overline{\mathcal{FM}_{\rm fin}(n)}$,

where $T_{\rm fin}$ = tracial states that factor through finite dim. C^*-alg.

Thm (Haagerup-M '15) Connes Embedding Problem (CEP) has positive answer **iff** $\overline{\mathcal{FM}_{\rm fin}(n)} = \mathcal{FM}(n)$, $\forall n \geq 3$.

Question: What can we say about $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}$?

- (Exel–Loring '92): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ residually finite dim. (RFD)
- (Blackadar '85): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ semi-projective.

The affine cont surj $\Phi\colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto T_\tau$, satisfies

- $\Phi(T_{\text{fin}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\text{fin}}(n)$,
- $\Phi(\overline{T_{\text{fin}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}) = \overline{\mathcal{FM}_{\text{fin}}(n)}$,

where $T_{\text{fin}} =$ tracial states that factor through finite dim. C^*-alg.

Thm (Haagerup-M '15) Connes Embedding Problem (CEP) has positive answer **iff** $\overline{\mathcal{FM}_{\text{fin}}(n)} = \mathcal{FM}(n)$, $\forall n \geq 3$.

Question: What can we say about $\overline{T_{\text{fin}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}$?

- (Exel–Loring '92): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ residually finite dim. (RFD)
- (Blackadar '85): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ semi-projective.

The affine cont surj $\Phi\colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto T_\tau$, satisfies

- $\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n)$,
- $\Phi(\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}) = \overline{\mathcal{FM}_{\rm fin}(n)}$,

where $T_{\rm fin}$ = tracial states that factor through finite dim. C^*-alg.

Thm (Haagerup-M '15) Connes Embedding Problem (CEP) has positive answer **iff** $\overline{\mathcal{FM}_{\rm fin}(n)} = \mathcal{FM}(n)$, $\forall n \geq 3$.

Question: What can we say about $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}$?

- (Exel–Loring '92): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ residually finite dim. (RFD)
- (Blackadar '85): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ semi-projective.

In general, given $A =$ (sep) unital tracial C^*-algebra, we have inclusions:

$$T_{\rm fin}(A) \subseteq \overline{T_{\rm fin}(A)} \subseteq T_{\rm qd}(A) \subseteq T_{\rm am}(A) \subseteq T_{\rm hyp}(A) \subseteq T(A),$$

where $T_{\rm qd}(A) =$ quasi-diagonal traces, $T_{\rm am}(A) =$ amenable (liftable) traces, $T_{\rm hyp}(A) =$ hyperlinear traces (i.e., traces τ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

- CEP pos answer **iff** $T_{\rm hyp}(A) = T(A)$, for all C^*-alg A.

Reformulation of CEP: For all sep. unital tracial C^*-algs (A, τ), there is a unital trace- preserving *-hom $\varphi\colon A \to \prod_{n=1}^{\infty} M_{k_n}/I^\omega$, for some $k_n \geq 1$.

- (N. Brown '06): $\exists$ exact RFD C^*-alg A s.t. $T_{\rm am}(A) \neq T_{\rm hyp}(A)$.
- Open if $T_{\rm qd}(A) = T_{\rm am}(A)$. Strong pos results: Tikuisis-Winter-White, Schafhauser, Gabe.
- A (weakly) semi-projective $\implies$ $\overline{T_{\rm fin}(A)} = T_{\rm qd}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD C^*-alg A s.t. $\overline{T_{\rm fin}(A)} \neq T_{\rm qd}(A)$.

Thm (Rørdam-M '20): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

In general, given $A =$ (sep) unital tracial C^*-algebra, we have inclusions:

$$T_{\rm fin}(A) \subseteq \overline{T_{\rm fin}(A)} \subseteq T_{\rm qd}(A) \subseteq T_{\rm am}(A) \subseteq T_{\rm hyp}(A) \subseteq T(A),$$

where $T_{\rm qd}(A) =$ quasi-diagonal traces, $T_{\rm am}(A) =$ amenable (liftable) traces, $T_{\rm hyp}(A) =$ hyperlinear traces (i.e., traces τ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

- CEP pos answer **iff** $T_{\rm hyp}(A) = T(A)$, for all C^*-alg A.

Reformulation of CEP: For all sep. unital tracial C^*-algs (A, τ), there is a unital trace- preserving *-hom $\varphi\colon A \to \prod_{n=1}^{\infty} M_{k_n}/I^\omega$, for some $k_n \geq 1$.

- (N. Brown '06): $\exists$ exact RFD C^*-alg A s.t. $T_{\rm am}(A) \neq T_{\rm hyp}(A)$.
- Open if $T_{\rm qd}(A) = T_{\rm am}(A)$. Strong pos results: Tikuisis-Winter-White, Schafhauser, Gabe.
- A (weakly) semi-projective $\implies \overline{T_{\rm fin}(A)} = T_{\rm qd}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD C^*-alg A s.t. $\overline{T_{\rm fin}(A)} \neq T_{\rm qd}(A)$.

Thm (Rørdam-M '20): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

In general, given $A =$ (sep) unital tracial C^*-algebra, we have inclusions:

$$T_{\rm fin}(A) \subseteq \overline{T_{\rm fin}(A)} \subseteq T_{\rm qd}(A) \subseteq T_{\rm am}(A) \subseteq T_{\rm hyp}(A) \subseteq T(A),$$

where $T_{\rm qd}(A) =$ quasi-diagonal traces, $T_{\rm am}(A) =$ amenable (liftable) traces, $T_{\rm hyp}(A) =$ hyperlinear traces (i.e., traces τ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

- CEP pos answer **iff** $T_{\rm hyp}(A) = T(A)$, for all C^*-alg A.

Reformulation of CEP: For all sep. unital tracial C^*-algs (A, τ), there is a unital trace- preserving *-hom $\varphi\colon A \to \prod_{n=1}^\infty M_{k_n}/I^\omega$, for some $k_n \geq 1$.

- (N. Brown '06): $\exists$ exact RFD C^*-alg A s.t. $T_{\rm am}(A) \neq T_{\rm hyp}(A)$.
- Open if $T_{\rm qd}(A) = T_{\rm am}(A)$. Strong pos results: Tikuisis-Winter-White, Schafhauser, Gabe.
- A (weakly) semi-projective $\Longrightarrow$ $\overline{T_{\rm fin}(A)} = T_{\rm qd}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD C^*-alg A s.t. $\overline{T_{\rm fin}(A)} \neq T_{\rm qd}(A)$.

Thm (Rørdam-M '20): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

In general, given $A =$ (sep) unital tracial C^*-algebra, we have inclusions:

$$T_{\mathrm{fin}}(A) \subseteq \overline{T_{\mathrm{fin}}(A)} \subseteq T_{\mathrm{qd}}(A) \subseteq T_{\mathrm{am}}(A) \subseteq T_{\mathrm{hyp}}(A) \subseteq T(A),$$

where $T_{\mathrm{qd}}(A) =$ quasi-diagonal traces, $T_{\mathrm{am}}(A) =$ amenable (liftable) traces, $T_{\mathrm{hyp}}(A) =$ hyperlinear traces (i.e., traces τ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

- CEP pos answer **iff** $T_{\mathrm{hyp}}(A) = T(A)$, for all C^*-alg A.

Reformulation of CEP: For all sep. unital tracial C^*-algs (A, τ), there is a unital trace- preserving *-hom $\varphi \colon A \to \prod_{n=1}^{\infty} M_{k_n}/I^\omega$, for some $k_n \geq 1$.

- (N. Brown '06): $\exists$ exact RFD C^*-alg A s.t. $T_{\mathrm{am}}(A) \neq T_{\mathrm{hyp}}(A)$.
- Open if $T_{\mathrm{qd}}(A) = T_{\mathrm{am}}(A)$. Strong pos results: Tikuisis-Winter-White, Schafhauser, Gabe.
- A (weakly) semi-projective $\implies$ $\overline{T_{\mathrm{fin}}(A)} = T_{\mathrm{qd}}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD C^*-alg A s.t. $\overline{T_{\mathrm{fin}}(A)} \neq T_{\mathrm{qd}}(A)$.

Thm (Rørdam-M '20): $\overline{T_{\mathrm{fin}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\mathrm{hyp}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

In general, given $A =$ (sep) unital tracial C^*-algebra, we have inclusions:

$$T_{\rm fin}(A) \subseteq \overline{T_{\rm fin}(A)} \subseteq T_{\rm qd}(A) \subseteq T_{\rm am}(A) \subseteq T_{\rm hyp}(A) \subseteq T(A),$$

where $T_{\rm qd}(A) =$ quasi-diagonal traces, $T_{\rm am}(A) =$ amenable (liftable) traces, $T_{\rm hyp}(A) =$ hyperlinear traces (i.e., traces τ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

- CEP pos answer **iff** $T_{\rm hyp}(A) = T(A)$, for all C^*-alg A.

Reformulation of CEP: For all sep. unital tracial C^*-algs (A, τ), there is a unital trace- preserving *-hom $\varphi\colon A \to \prod_{n=1}^\infty M_{k_n}/I^\omega$, for some $k_n \geq 1$.

- (N. Brown '06): $\exists$ exact RFD C^*-alg A s.t. $T_{\rm am}(A) \neq T_{\rm hyp}(A)$.
- Open if $T_{\rm qd}(A) = T_{\rm am}(A)$. Strong pos results: Tikuisis-Winter-White, Schafhauser, Gabe.
- A (weakly) semi-projective $\implies$ $\overline{T_{\rm fin}(A)} = T_{\rm qd}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD C^*-alg A s.t. $\overline{T_{\rm fin}(A)} \neq T_{\rm qd}(A)$.

Thm (Rørdam-M '20): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

In general, given $A =$ (sep) unital tracial C^*-algebra, we have inclusions:

$$T_{\rm fin}(A) \subseteq \overline{T_{\rm fin}(A)} \subseteq T_{\rm qd}(A) \subseteq T_{\rm am}(A) \subseteq T_{\rm hyp}(A) \subseteq T(A),$$

where $T_{\rm qd}(A) =$ quasi-diagonal traces, $T_{\rm am}(A) =$ amenable (liftable) traces, $T_{\rm hyp}(A) =$ hyperlinear traces (i.e., traces τ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

- CEP pos answer **iff** $T_{\rm hyp}(A) = T(A)$, for all C^*-alg A.

Reformulation of CEP: For all sep. unital tracial C^*-algs (A, τ), there is a unital trace- preserving *-hom $\varphi\colon A \to \prod_{n=1}^\infty M_{k_n}/I^\omega$, for some $k_n \geq 1$.

- (N. Brown '06): $\exists$ exact RFD C^*-alg A s.t. $T_{\rm am}(A) \neq T_{\rm hyp}(A)$.
- Open if $T_{\rm qd}(A) = T_{\rm am}(A)$. Strong pos results: Tikuisis-Winter-White, Schafhauser, Gabe.
- A (weakly) semi-projective $\implies$ $\overline{T_{\rm fin}(A)} = T_{\rm qd}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD C^*-alg A s.t. $\overline{T_{\rm fin}(A)} \neq T_{\rm qd}(A)$.

Thm (Rørdam-M '20): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

In general, given $A =$ (sep) unital tracial C^*-algebra, we have inclusions:

$$T_{\rm fin}(A) \subseteq \overline{T_{\rm fin}(A)} \subseteq T_{\rm qd}(A) \subseteq T_{\rm am}(A) \subseteq T_{\rm hyp}(A) \subseteq T(A),$$

where $T_{\rm qd}(A) =$ quasi-diagonal traces, $T_{\rm am}(A) =$ amenable (liftable) traces, $T_{\rm hyp}(A) =$ hyperlinear traces (i.e., traces τ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

- CEP pos answer **iff** $T_{\rm hyp}(A) = T(A)$, for all C^*-alg A.

Reformulation of CEP: For all sep. unital tracial C^*-algs (A, τ), there is a unital trace- preserving *-hom $\varphi\colon A \to \prod_{n=1}^\infty M_{k_n}/I^\omega$, for some $k_n \geq 1$.

- (N. Brown '06): $\exists$ exact RFD C^*-alg A s.t. $T_{\rm am}(A) \neq T_{\rm hyp}(A)$.
- Open if $T_{\rm qd}(A) = T_{\rm am}(A)$. Strong pos results: Tikuisis-Winter-White, Schafhauser, Gabe.
- A (weakly) semi-projective $\implies$ $\overline{T_{\rm fin}(A)} = T_{\rm qd}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD C^*-alg A s.t. $\overline{T_{\rm fin}(A)} \neq T_{\rm qd}(A)$.

Thm (Rørdam-M '20): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

Thm (Rørdam-M): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

Cor: CEP pos **iff** $T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C})) = T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $\forall\, n \geq 3$.

Further results: Let A be a unital C^*-algebra.

- If $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, then A generated by n^2 elem.
- If A gen by $n-1$ unitaries, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$.

As unital separable $\mathcal{Z}$-stable C*-alg are singly generated (Thiel-Winter), we deduce: If A is a simple unital inf dim AF-algebra, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, $n \geq 3$.

Thm (Rørdam-M): Each metrizable Choquet simplex is affinely homeo to a face of $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $n \geq 3$.

(Open) question: Is $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$ the Poulsen simplex?

Thm (Rørdam-M): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

Cor: CEP pos **iff** $T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C})) = T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $\forall\, n \geq 3$.

Further results: Let A be a unital C^*-algebra.

- If $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, then A generated by n^2 elem.
- If A gen by $n-1$ unitaries, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$.

As unital separable $\mathcal{Z}$-stable C*-alg are singly generated (Thiel-Winter), we deduce: If A is a simple unital inf dim AF-algebra, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, $n \geq 3$.

Thm (Rørdam-M): Each metrizable Choquet simplex is affinely homeo to a face of $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $n \geq 3$.

(Open) question: Is $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$ the Poulsen simplex?

Thm (Rørdam-M): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

Cor: CEP pos **iff** $T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C})) = T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $\forall\, n \geq 3$.

Further results: Let A be a unital C^*-algebra.

- If $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, then A generated by n^2 elem.
- If A gen by $n-1$ unitaries, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$.

As unital separable $\mathcal{Z}$-stable C*-alg are singly generated (Thiel-Winter), we deduce: If A is a simple unital inf dim AF-algebra, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, $n \geq 3$.

Thm (Rørdam-M): Each metrizable Choquet simplex is affinely homeo to a face of $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $n \geq 3$.

(Open) question: Is $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$ the Poulsen simplex?

Thm (Rørdam-M): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

Cor: CEP pos **iff** $T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C})) = T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $\forall\ n \geq 3$.

Further results: Let A be a unital C^*-algebra.

- If $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, then A generated by n^2 elem.
- If A gen by $n-1$ unitaries, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$.

As unital separable $\mathcal{Z}$-stable C*-alg are singly generated (Thiel-Winter), we deduce: If A is a simple unital inf dim AF-algebra, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, $n \geq 3$.

Thm (Rørdam-M): Each metrizable Choquet simplex is affinely homeo to a face of $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $n \geq 3$.

(Open) question: Is $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$ the Poulsen simplex?

Further connections: Analysis of quantum correlations and CEP

- $\Gamma := \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$, n copies, $n, k \geq 2$.
- $C^*(\Gamma) = C^*(q_{j,x} \mid q_{j,x} = q_{j,x}^* = q_{j,x}^2, \sum_{j=1}^k q_{j,x} = 1)$.

(Schafhauser, AIM '21): For all $n, k \geq 2$, we have

$$\overline{T_{\rm fin}(C^*(\Gamma))} = T_{\rm hyp}(C^*(\Gamma)).$$

Definition: A "correlation" $[(p(i,j \mid x,y)]$ is *synchronous* if $\forall\, 1 \leq x \leq n$, $p(i,j \mid x,x) = 0$ whenever $i \neq j$.

Theorem (Paulsen-Severini-Stalke-Todorov-Winter '16):

$$\begin{aligned} C_{qc}^s(n,k) &= \Big\{ \big[\tau(q_{j,x}q_{i,y})\big]_{(i,x;j,y)} \mid \tau \in T(C^*(\Gamma)) \Big\} \\ C_q^s(n,k) &= \Big\{ \big[\tau(q_{j,x}q_{i,y})\big]_{(i,x;j,y)} \mid \tau \in T_{\rm fin}(C^*(\Gamma)) \Big\}. \end{aligned}$$

Further connections: Analysis of quantum correlations and CEP

- $\Gamma := \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$, n copies, $n, k \geq 2$.
- $C^*(\Gamma) = C^*(q_{j,x} \mid q_{j,x} = q_{j,x}^* = q_{j,x}^2,\ \sum_{j=1}^k q_{j,x} = 1)$.

(Schafhauser, AIM '21): For all $n, k \geq 2$, we have

$$\overline{T_{\mathrm{fin}}(C^*(\Gamma))} = T_{\mathrm{hyp}}(C^*(\Gamma))).$$

Definition: A "correlation" $[(p(i,j \mid x,y)]$ is *synchronous* if $\forall\, 1 \leq x \leq n$, $p(i,j \mid x,x) = 0$ whenever $i \neq j$.

Theorem (Paulsen-Severini-Stalke-Todorov-Winter '16):

$$\begin{aligned} C_{qc}^s(n,k) &= \left\{ \left[\tau(q_{j,x}q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T(C^*(\Gamma)) \right\} \\ C_q^s(n,k) &= \left\{ \left[\tau(q_{j,x}q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T_{\mathrm{fin}}(C^*(\Gamma)) \right\}. \end{aligned}$$

Further connections: Analysis of quantum correlations and CEP

- $\Gamma := \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$, n copies, $n, k \geq 2$.
- $C^*(\Gamma) = C^*(q_{j,x} \mid q_{j,x} = q_{j,x}^* = q_{j,x}^2, \ \sum_{j=1}^k q_{j,x} = 1)$.

(Schafhauser, AIM '21): For all $n, k \geq 2$, we have

$$\overline{T_{\mathrm{fin}}(C^*(\Gamma))} = T_{\mathrm{hyp}}(C^*(\Gamma)).$$

Definition: A "correlation" $[(p(i,j \mid x,y)]$ is *synchronous* if $\forall\, 1 \leq x \leq n$, $p(i,j \mid x,x) = 0$ whenever $i \neq j$.

Theorem (Paulsen-Severini-Stalke-Todorov-Winter '16):

$$\begin{aligned} C_{qc}^s(n,k) &= \left\{ \left[\tau(q_{j,x}q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T(C^*(\Gamma)) \right\} \\ C_q^s(n,k) &= \left\{ \left[\tau(q_{j,x}q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T_{\mathrm{fin}}(C^*(\Gamma)) \right\}. \end{aligned}$$

Set $C_{qa}(n,k) := \overline{C_q(n,k)}$.

(Kim-Paulsen-Schafhauser '17): $C^s_{qa}(n,k) = \overline{C^s_q(n,k)}$.

Theorem (Kim-Paulsen-Schafhauser '17, Ozawa '13): TFAE

(1) Connes Embedding Problem has positive answer.

(2) $C^s_{qa}(n,k) = C^s_{qc}(n,k)$, $\forall n, k \geq 2$.

(3) Tsirelson's conjecture is true, i.e., $C_{qa}(n,k) = C_{qc}(n,k)$, $\forall n, k \geq 2$.

Theorem (Fritz/Junge et. al. '09):

- $\mathcal{C}_{qa}(n,k) = \left\{ \left[\varphi(q_{j,x} \otimes q_{i,y}) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \right\}$.
- $\mathcal{C}_{qc}(n,k) = \left\{ \left[\varphi(q_{j,x} \otimes q_{i,y}) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \right\}$.

- $A := C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ is RFD $[\Rightarrow S_{\text{fin}}(A) \overset{\text{dense}}{\subseteq} S(A)]$.

Set $C_{qa}(n,k) := \overline{C_q(n,k)}$.

(Kim-Paulsen-Schafhauser '17): $C_{qa}^s(n,k) = \overline{C_q^s(n,k)}$.

Theorem (Kim-Paulsen-Schafhauser '17, Ozawa '13): TFAE

(1) Connes Embedding Problem has positive answer.

(2) $C_{qa}^s(n,k) = C_{qc}^s(n,k)$, $\forall n,k \geq 2$.

(3) Tsirelson's conjecture is true, i.e., $C_{qa}(n,k) = C_{qc}(n,k)$, $\forall n,k \geq 2$.

Theorem (Fritz/Junge et. al. '09):

- $\mathcal{C}_{qa}(n,k) = \left\{ \left[\varphi(q_{j,x} \otimes q_{i,y}) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \right\}$.
- $\mathcal{C}_{qc}(n,k) = \left\{ \left[\varphi(q_{j,x} \otimes q_{i,y}) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \right\}$.

- $A := C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ is RFD $[\Rightarrow S_{\text{fin}}(A) \overset{\text{dense}}{\subseteq} S(A)]$.

Set $C_{qa}(n,k) := \overline{C_q(n,k)}$.

(Kim-Paulsen-Schafhauser '17): $C^s_{qa}(n,k) = \overline{C^s_q(n,k)}$.

Theorem (Kim-Paulsen-Schafhauser '17, Ozawa '13): TFAE

(1) Connes Embedding Problem has positive answer.

(2) $C^s_{qa}(n,k) = C^s_{qc}(n,k)$, $\forall n,k \geq 2$.

(3) Tsirelson's conjecture is true, i.e., $C_{qa}(n,k) = C_{qc}(n,k)$, $\forall n,k \geq 2$.

Theorem (Fritz/Junge et. al. '09):

- $\mathcal{C}_{qa}(n,k) = \left\{ \left[\varphi(q_{j,x} \otimes q_{i,y}) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \right\}$.
- $\mathcal{C}_{qc}(n,k) = \left\{ \left[\varphi(q_{j,x} \otimes q_{i,y}) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \right\}$.

- $A := C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ is RFD $[\Rightarrow S_{\text{fin}}(A) \overset{\text{dense}}{\subseteq} S(A)]$.

▶ **Posted on arXiv, Jan 2020:** $MIP^* = RE$, Ji, Natarajan, Vidick, Wright, Yuen, 165 pp.

Proving that the complexity class MIP* (quantum version of complexity class MIP=languages with a Multiprover Interactive Proof) contains an undecidable language, they conclude that Tsirelson's Conjecture is **false**!

New version (with corrections) 206 pp., posted on arXiv, Sept 2020.

▶ Further recent applications of/connections to factorizability:

Gangbo-Jeckel-Nam-Shlyakhtenko, May 2021: "Duality for optimal couplings in free probability".

- **Posted on arXiv, Jan 2020:** $MIP^* = RE$, Ji, Natarajan, Vidick, Wright, Yuen, 165 pp.

Proving that the complexity class MIP* (quantum version of complexity class MIP=languages with a Multiprover Interactive Proof) contains an undecidable language, they conclude that Tsirelson's Conjecture is **false**!

New version (with corrections) 206 pp., posted on arXiv, Sept 2020.

- Further recent applications of/connections to factorizability:

Gangbo-Jeckel-Nam-Shlyakhtenko, May 2021: "Duality for optimal couplings in free probability".