# Wold decomposition on self-similar graphs

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# Background and Motivation

**Theorem** (Wold-decomposition Theorem)

*Every isometry $V \in \mathcal{B}(\mathcal{H})$ can be decomposed as $V = U \oplus S^{(\alpha)}$ where $U$ is unitary, and $S^{(\alpha)}$ is a direct sum of $\alpha$-copies of the unilateral shift.*

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The Wold-decomposition theorem is an extremely useful theorem in operator theory. For example, it leads to Coburn's theorem:

**Theorem** (Coburn, 1967)

*Let* $V$ *be any isometry that is not unitary. Then* $C^*(V) = \overline{\mathrm{Alg}}\{V, V^*\}$ *is isometrically isomorphic to* $C^*(S)$.

# Background and Motivation

Generalizations?

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**Theorem** (Słociński 1980)

*A pair of doubly commuting isometries $S_1, S_2$ decomposes into a direct sum of four components: unitary-unitary, unitary-pure, pure-unitary, and pure-pure. In particular, the pure-pure component is equivalent to a direct sum of the bishift on $\ell^2(\mathbb{N}^2)$.*

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**Goal**: Study Wold decomposition for "nice" isometric representations of semigroups.

# Background and Motivation

## Definition

*The odometer semigroup $\mathbb{O}_n$ is the unital semigroup generated by $n$ free generators $v_1, \cdots, v_n$ and an additional generator $w$ such that*

$$wv_k = \begin{cases} v_{k+1}, & \text{if } 1 \le k \le n-1; \\ v_1 w, & \text{if } k = n. \end{cases}$$

## Definition

*A Nica-covariant isometric representation of $\mathbb{O}_n$ is defined to be an isometry $W$ and a row isometry $\{V_1, \cdots, V_n\}$ such that $W^* V_1 = V_n W^*$ and*

$$WV_k = \begin{cases} V_{k+1}, & \text{if } 1 \le k \le n-1; \\ V_1 W, & \text{if } k = n. \end{cases}$$

# Background and Motivation

**Theorem** (L. 2021)

*An isometric Nica-covariant representation* $\{W, V_1, \cdots, V_n\}$ *of* $\mathbb{O}_n$ *decomposes into a direct sum of four components: unitary-row unitary, unitary-pure row isometry, pure-row unitary, and pure-pure row isometry.*

*In particular, the pure-pure row isometry component is equivalent to a direct sum of the left-regular representation of* $\mathbb{O}_n$ *on* $\ell^2(\mathbb{O}_n)$.

# Self-Similar Graph

## Definition

*Let $E$ be a directed graph. An automorphism of $E$ is a bijective map $\pi : E^0 \to E^0$ and $E^1 \to E^1$ such that*

$$\pi \circ r = r \circ \pi \text{ and } \pi \circ s = s \circ \pi.$$

A $P$-action on $E$ is defined as a homomorphism $P \to \mathrm{Aut}(E)$. A self-similar action also encodes an "$E$-action on $P$".

# Self-Similar Graph

## Definition

*Let $E$ be a directed graph and $P$ be a semigroup. A a self-similar action of $P$ on $E$ consists of*

1. *A $P$-action map denoted by $P \times E^* \to E^*$, $(p, \mu) \mapsto p \cdot \mu$*
2. *A restriction map denoted by $P \times E^* \to P$, $(p, \mu) \mapsto p|_\mu$*

*In addition of the following properties:*

1. $p \cdot (\mu\nu) = (p \cdot \mu)(p|_\mu \cdot \nu)$ *for all* $p \in P, \mu, \nu \in E^*$ *with* $s(\mu) = r(\nu)$.
2. $p|_v = p$ *for all* $p \in P, v \in E^0$;
3. $p|_{\mu\nu} = p|_\mu|_\nu$ *for all* $p \in P, \mu, \nu \in E^*$ *with* $s(\mu) = r(\nu)$;
4. $1_P|_\mu = 1_P$ *for all* $\mu \in E^*$;
5. $(pq)|_\mu = p|_{q\cdot\mu} q|_\mu$ *for all* $p, q \in P, \mu \in E^*$.

*In this case, $E$ is also called a self-similar graph over $P$, or $(P, E)$ a self-similar graph.*

# Self-Similar Graph

Every self-similar graph is associated with a semigroupoid structure.

## Definition

*Let $(P, E)$ be a self-similar graph. We define the self-similar product semigroupoid to be $E^* \bowtie P = \{(\mu, p) : \mu \in E^*, p \in P\}$, where the multiplication is given by:*

$$(\mu, p) \cdot (\nu, q) = (\mu(p \cdot \nu), p|_\nu q), \ \textit{if } s(\mu) = r(p \cdot \nu).$$

Note: In particular, if $E$ is a single vertex graph with $n$ edges, then $E^* \bowtie P$ becomes a semigroup, which coincides with a Zappa-Szép product semigroup $\mathbb{F}_n^+ \bowtie P$ (Brownlowe et. al.).

# Self-Similar Graph

## Example

*The odometer semigroup $\mathbb{O}_n$ is a self-similar product semigroup where $E$ has single vertex and $n$ edges, and $P = \mathbb{N}$. The $\mathbb{N}$-action map is defined as $1 \cdot e_i = e_{i+1}$ and the $\mathbb{N}$-restriction map is given by*

$$1|_{e_i} = \begin{cases} 0, & \text{if } 1 \leq i \leq n-1, \\ 1, & \text{if } i = n. \end{cases}$$

# Self-Similar Graph

## Example

*The Baumslag-Solitar monoids* $\mathrm{BS}^+(n,m)$ *is defined as*

$$\mathrm{BS}^+(n,m) = \langle a, b | a^n b = b a^m \rangle.$$

*Notice that the elements* $\{b, ab, \cdots, a^{n-1}b\}$ *are free in the semigroup, and therefore generate a copy of* $\mathbb{F}_n^+$. *We can realize it as a self-similar product semigroup where* $E$ *has single vertex and* $n$ *edges, and* $P = \mathbb{N}$. *The* $\mathbb{N}$*-action map is defined as* $1 \cdot e_i = e_{i+1}$ *and the* $\mathbb{N}$*-restriction map is given by*

$$1|_{e_i} = \begin{cases} 0, & \text{if } 1 \le i \le n-1, \\ 1, & \text{if } i = n. \end{cases}$$

*Here, we can identify* $e_i = a^{i-1}b$ *in the graph* $E$ *and* $1 = a$ *in* $\mathbb{N}$.

# Representations of Self-Similar Graph

## Definition

*Let $E$ be a directed graph. A Toeplitz-Cuntz-Kreiger (TCK) family is given by* $\mathcal{S} := \{S_\mu : \mu \in E\}$ *such that:*

1. $\{S_v\}$ *are pairwise orthogonal projections.*
2. *For each* $e \in E^1$, $S_e^* S_e = S_{s(e)}$.
3. *For each* $v \in E^0$,
$$\sum_{r(e)=v} S_e S_e^* \leq S_v.$$

*We say this family is a Cuntz-Kreiger (CK) family if the last condition is replaced by*

$$\sum_{r(e)=v} S_e S_e^* = S_v, \text{ for all } v \in E^0. \tag{CK}$$

# Representations of Self-Similar Graph

**Definition**

*A Toeplitz representation of a self-similar graph* $(P, E)$ *consists of an isometric representation* $\mathcal{V}$ *of* $P$ *and a TCK family* $\mathcal{S}$ *of* $E$*, such that*

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$$V_p S_\mu = S_{p \cdot \mu} V_{p|_\mu} \tag{1}$$

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1. *For all* $p \in P$ *and* $\mu \in E^*$,
$$V_p S_\mu = S_{p\cdot\mu} V_{p|_\mu} \tag{1}$$
2. *For all* $p \in P$ *and* $\mu \in E^*$,
$$V_p^* S_{p\cdot\mu} = S_\mu V_{p|_\mu}^* \tag{2}$$

# "Left-regular" representations

An archetypal example of a Toeplitz representation is the "left-regular" representation.

### Example

*Consider* $\mathcal{K} = \ell^2(E^* \times P)$ *where we use* $\{\delta_{\mu,p} : \mu \in E^*, p \in P\}$ *to denote its orthonormal basis. For each* $q \in P$*, define*

$$V_q \delta_{\mu,p} = \delta_{q\cdot\mu, q|_\mu p}.$$

*For each* $\nu \in E^i$ $(i = 0, 1)$*, define*

$$S_\nu \delta_{\mu,p} = \begin{cases} \delta_{\nu\mu,p}, & \textit{if } s(\nu) = r(\mu), \\ 0, & \textit{if otherwise.} \end{cases}$$

*Then* $(\mathcal{S}, \mathcal{V})$ *defines a Toeplitz representation of the self-similar graph.*

**Note**: This representation is not irreducible.

# “Left-regular” representations - continued

## Example

*For each* $v \in E^0$, *let*

$$\mathcal{K}_v = \overline{\operatorname{span}}\{S_\mu V_p \delta_{v,e} : \mu \in E^*, p \in P\} = \overline{\operatorname{span}}\{\delta_{\mu,p} : \mu \in E^*, p \in P, s(\mu) = p \cdot v\}.$$

*One can verify that* $\mathcal{K}_v$ *reduces the left regular representation* $(\mathcal{V}, \mathcal{S})$. *We shall use the notation* $(\lambda_v^P, \lambda_v^E)$ *to denote the restriction of* $(\mathcal{V}, \mathcal{S})$ *on* $\mathcal{K}_v$. *A representation of a self-similar graph is called left regular if it is a direct sum of amplifications of these* $\lambda$.

# Wold Decomposition

The Wold decomposition for TCK family has been known in the literature.

**Theorem**

*Let $\mathcal{S}$ be a TCK family of a directed graph $E$ on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ decomposes as $\mathcal{H} = \mathcal{H}_C \oplus \left(\bigoplus_{v\in E^0} \mathcal{H}_v\right)$, under which $\mathcal{S}$ is decomposed as $\mathcal{S} = \mathcal{T} \oplus \left(\bigoplus_{v\in E^0} L_{E,v}^{(\alpha_v)}\right)$. Here,*

- *On $\mathcal{H}_C$, $T$ is a CK family.*
- *On each $\mathcal{H}_v$, the restriction is an amplification of the "shift operators".*

*Moreover, let $\mathcal{W}_v = (S_v - \sum_{r(e)=v} S_e S_e^*)\mathcal{H}$. We have $\mathcal{H}_v = \oplus_{\mu\in E^*} S_\mu \mathcal{W}_v$, and $\alpha_v = \dim \mathcal{W}_v$.*

# Wold Decomposition

For each $v \in E^0$, define its orbit $O_v = \{p \cdot v : p \in P\}$. Since we assumed $E$ is finite and $P$ act by automorphisms, $\{O_v\}$ is a partition of $E^0$ (and defines an equivalence relation).

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**Theorem** (L.-Yang)

*Let $(\mathcal{V}, \mathcal{S})$ be a Toeplitz representation of a self-similar graph $(P, E)$ on a Hilbert space $\mathcal{H}$. Let $\mathcal{W}_v$, $\mathcal{H}_v$, $\mathcal{H}_C$ as defined previously. Then*

1. *$\mathcal{H}_C$ reduces $\mathcal{V}$.*

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4. *For each $O_v$, $\mathcal{H}_{[v]} = \bigoplus_{u \in O_v} \mathcal{H}_u$ reduces $\mathcal{V}$.*

Special case when $P$ is a group?

# Wold Decomposition

We would like to study how the Wold decomposition of $\mathcal{V}$ on $P$ interacts with the Wold decomposition of $E$. However, Wold decomposition on general semigroups is poorly studied. We focus on the case when $P = \mathbb{N}$, so $\mathcal{V}$ is determined by a single isometry $V$.
**Assumption**. For every $e \in E^1$, there is $m \in \mathbb{N}$ such that $m|_e \neq 0$.

## Proposition

*Let $(\mathcal{V}, \mathcal{S})$ be a Toeplitz representation of a self-similar graph $(\mathbb{N}, E)$ on a Hilbert space $\mathcal{H}$. Suppose the Wold decomposition of $V$ is given by $\mathcal{H} = \mathcal{H}^U \oplus \mathcal{H}^S$. Then $\mathcal{H}^U$ and $\mathcal{H}^S$ both reduces $\mathcal{S}$.*

# Wold Decomposition - Main Result

**Theorem** (Yang-L.)

*Let $(\mathcal{V}, \mathcal{S})$ be a Toeplitz representation of a self-similar graph $(\mathbb{N}, E)$ on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ decomposes as a direct sum of reducing subspaces:*

$$\mathcal{H} = \mathcal{H}_C^U \oplus \mathcal{H}_C^S \oplus \left( \bigoplus_{O_v \in E^0/\sim} \mathcal{H}_{[v]}^U \right) \oplus \left( \bigoplus_{v \in E^0} \mathcal{H}_v^S \right).$$

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***1. On*** *$\mathcal{H}_C^U$, $\mathcal{V}$ is a unitary and $\mathcal{S}$ is a CK family.*

# Wold Decomposition - Main Result, continued

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***2. On** $\mathcal{H}_C^S$, $\mathcal{V}$ is pure and $\mathcal{S}$ is a CK family.*

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***3. On each*** $\mathcal{H}_{[v]}^U$, *$\mathcal{V}$ is a unitary and $\mathcal{S}$ is a direct sum of some amplifications of $\bigoplus_{u \in \Omega_v} L_{E,u}$. The multiplicity equals the dimension of $\mathcal{W}_v = (S_v - \sum_{r(e)=v})\mathcal{H}^U$. Moreover, $\mathcal{V}$ is unitary on $\oplus_{u \in O_v} \mathcal{W}_u$.*

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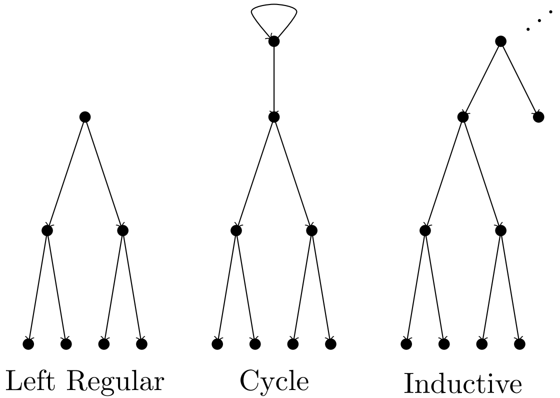
*4.* ***On each*** $\mathcal{H}_v^S$*,* $(\mathcal{V}, \mathcal{S})$ *is unitarily equivalent to an amplification of the left regular representation* $(\lambda_v^P, \lambda_v^E)^{(\alpha_v)}$*, where* $\alpha_v$ *is the dimension of the space* $\mathcal{W}_{P,v} = \ker V^* \cap (S_v - \sum_{r(e)=u} S_e S_e^*)\mathcal{H}_v^S$.

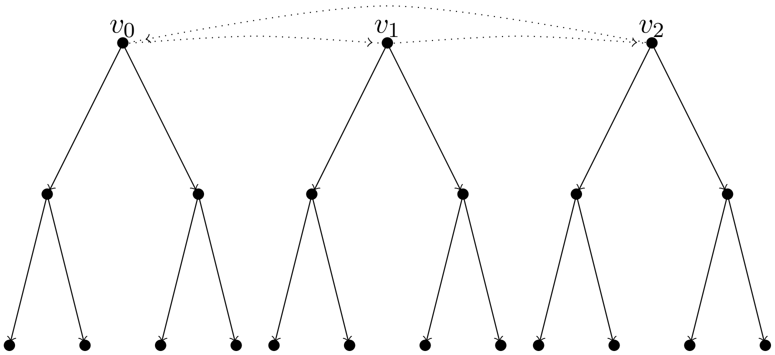
# Atomic representations

Davidson and Pitts first characterized atomic representations of row isometries. This has been extended to TCK families by Davidson-Doron-L.. There are three types:



# Atomic representations

**Unitary + Shift Type**:



# Atomic representations

**Unitary + Shift Type**:

**Proposition**

*Let $O_v$ be an orbit of $v \in E^0$, and suppose $(\mathcal{V}, \mathcal{S})$ is a Toeplitz representation on $\mathcal{H}_{[v]}^U = \bigoplus_{\mu \in E^*} S_\mu \mathcal{W}_{[v]}$ (i.e., $\mathcal{V}$ is unitary and $\mathcal{S}$ is an amplification of $\bigoplus_{u \in O_v} L_{E,u}$). Then $\mathcal{V}$ is uniquely determined by $\mathcal{V}|_{\mathcal{W}_{[v]}}$.*

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*Proof.* $V_p S_\mu h = S_{p\cdot\mu} V_{p|_\mu} h.$

# Atomic representations

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*Proof.* $V_p S_\mu h = S_{p \cdot \mu} V_{p|_\mu} h$.

## Corollary

*Let $\mathcal{S}$ be some amplification of $\bigoplus_{u \in O_v} L_{E,u}$ and $\mathcal{W}_{[v]}$ be its wandering space. Suppose $V_0$ is unitary on $\mathcal{W}_{[v]}$. Then there exists a unique Toeplitz representation $(\mathcal{V}, \mathcal{S})$ such that $\mathcal{V}|_{\mathcal{W}_{[v]}} = V_0$.*

# Atomic representations

**Unitary + CK Type**:

**Proposition**

*Suppose $\mathcal{S}$ is an inductive type atomic representation. Then there exists a unique $\mathcal{V}$ such that $(\mathcal{V}, \mathcal{S})$ is a Toeplitz representation. Moreover, this is unitary + CK type.*

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### Proposition

*Suppose $\mathcal{S}$ is an inductive type atomic representation. Then there exists a unique $\mathcal{V}$ such that $(\mathcal{V}, \mathcal{S})$ is a Toeplitz representation. Moreover, this is unitary + CK type.*

### Example

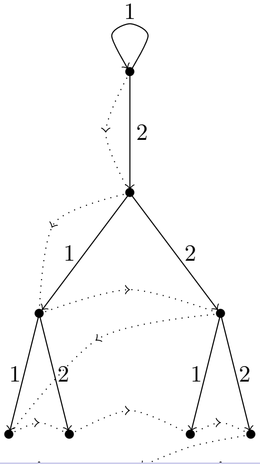
*Consider $\mathcal{S}$ be the shift on $\ell^2(E^\infty)$, which is an inductive type atomic representation. For each infinite path $\mu$, let $V_p\delta_\mu = \delta_{p\cdot\mu}$. Here,*

$$p \cdot (e_1 e_2 \cdots) = (p \cdot e_1)(p|_{e_1} \cdot e_2)(p|_{e_1 e_2} \cdot e_3) \cdots .$$

*Then $(\mathcal{V}, \mathcal{S})$ is unitary + CK type.*

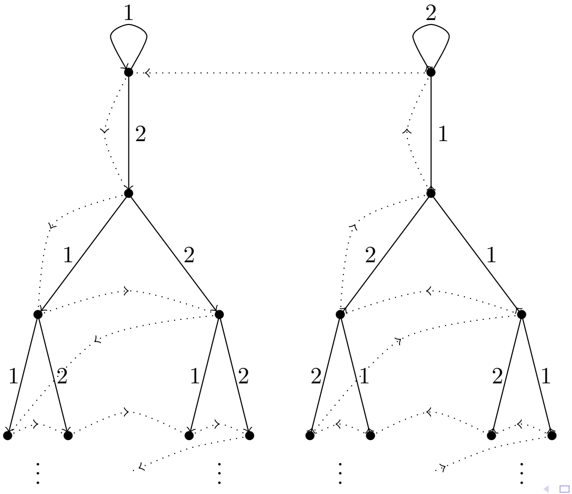
# Atomic representations

**Shift + CK Type**: If $\mathcal{S}$ is a cycle type where for each $e$ in the cycle, let $f$ be $1 \cdot f = e$. We must have $1|_f \neq 0$.



# Atomic representations

It is possible to have unitary + CK, where the CK family is cycle type.



Thank you