Remarks on properties of the Cuntz semigroup

Cristian Ivanescu

May 30, 2022

Table of Contents

Cuntz semigroup

$a, b \in M_\infty(A)_+$ then $a \precsim b$ if there is $v_n \in M_\infty(A)$ such that

$$v_n b v_n^* \to a$$

Cuntz semigroup

$a, b \in M_\infty(A)_+$ then $a \precsim b$ if there is $v_n \in M_\infty(A)$ such that

$$v_n b v_n^* \to a$$

$$a \backsim b \quad \text{if} \quad a \precsim b \quad \text{and} \quad b \precsim a$$

Cuntz semigroup

$a, b \in M_\infty(A)_+$ then $a \precsim b$ if there is $v_n \in M_\infty(A)$ such that

$$v_n b v_n^* \to a$$

$$a \backsim b \quad \text{if} \quad a \precsim b \quad \text{and} \quad b \precsim a$$

J. Cuntz (1978)

$$W(A) = (M_\infty(A)_+)/ \backsim$$

Cuntz semigroup

$a, b \in M_\infty(A)_+$ then $a \precsim b$ if there is $v_n \in M_\infty(A)$ such that

$$v_n b v_n^* \to a$$

$$a \backsim b \quad \text{if} \quad a \precsim b \quad \text{and} \quad b \precsim a$$

J. Cuntz (1978)

$$W(A) = (M_\infty(A)_+)/\backsim$$

$$Cu(A) = W(A \otimes \mathbb{K})$$

$$Cu(A) = (A \otimes \mathbb{K}_+)/\backsim$$

Cuntz semigroup

Initially to study Rank Functions on C*-alg. Lately, important role in the classification theory and study algebraic properties of C*-algebras.

$$Cu(\mathbb{C}) = Cu(M_n(\mathbb{C})) = \mathbb{N} \cup \infty$$

Cuntz semigroup

Initially to study Rank Functions on C*-alg. Lately, important role in the classification theory and study algebraic properties of C*-algebras.

$$Cu(\mathbb{C}) = Cu(M_n(\mathbb{C})) = \mathbb{N} \cup \infty$$

$$Cu(C[0,1]) = \{f : [0,1] \to \mathbb{N} \cup \infty, f = LSC\}$$

$$Cu(C[0,1]^3) = ??$$

- K-amenable groups (J. Cuntz 1983)

- K-amenable groups (J. Cuntz 1983)
- KK-amenable algebras (G. Skandalis 1988)

- K-amenable groups (J. Cuntz 1983)
- KK-amenable algebras (G. Skandalis 1988)
- K-amenable or KK-amenable does **NOT** imply nuclearity of the algebra

- K-amenable groups (J. Cuntz 1983)
- KK-amenable algebras (G. Skandalis 1988)
- K-amenable or KK-amenable does **NOT** imply nuclearity of the algebra

Def

A is **Cu-nuclear** if the canonical quotient map

$$\pi : A \otimes_{max} B \to A \otimes_{min} B$$

induces an isomorphism

$$Cu(A \otimes_{max} B) \cong Cu(A \otimes_{min} B)$$

for all C*-alg B.

-

Cu-nuclear and weakly Cu-nuclear

Def

A is **weakly Cu-nuclear** if

$$Cu(A \otimes_{max} B) \cong Cu(A \otimes_{min} B)$$

for all C*-alg B.

Cu-nuclear and weakly Cu-nuclear

Def

A is **weakly Cu-nuclear** if

$$Cu(A \otimes_{max} B) \cong Cu(A \otimes_{min} B)$$

for all C*-alg B.

Question

Is the class of **Cu-nuclear** a strict subset of **weakly Cu-nuclear** ?

Cu-nuclear and weakly Cu-nuclear

Def

A is **weakly Cu-nuclear** if

$$Cu(A \otimes_{max} B) \cong Cu(A \otimes_{min} B)$$

for all C*-alg B.

Question

Is the class of **Cu-nuclear** a strict subset of **weakly Cu-nuclear** ?

Question

weakly Cu-nuclear $\Rightarrow$ **nuclear** ?

Cu-nuclear

Theorem (I. Kučerovský)

- If A and B are simple C*-algebras

Cu-nuclear

Theorem (I. Kučerovský)

- If A and B are simple C*-algebras
- If A has the weakly Cu-nuclear property

Theorem (I. Kučerovský)

- If A and B are simple C*-algebras
- If A has the weakly Cu-nuclear property
- then

$$A \otimes_{min} B = A \otimes_{max} B$$

Cu-nuclear

Theorem (I. Kučerovský)

- If A and B are simple C*-algebras
- If A has the weakly Cu-nuclear property
- then

$$A \otimes_{min} B = A \otimes_{max} B$$

Question:

If A is simple and $A \otimes_{min} B = A \otimes_{max} B$ for all simple B then is A nuclear?

Cu-nuclear

Corollary

If a C*-algebra with finitely many ideals is weakly Cu-nuclear then it is exact and has the LLP.

Cu-nuclear

Corollary

If a C*-algebra with finitely many ideals is weakly Cu-nuclear then it is exact and has the LLP.

Corollary

If a C*-algebra has finitely many ideals, then weakly Cu-nuclear implies nuclear C*-algebra.

Cu-nuclear

Corollary

If a C*-algebra with finitely many ideals is weakly Cu-nuclear then it is exact and has the LLP.

Corollary

If a C*-algebra has finitely many ideals, then weakly Cu-nuclear implies nuclear C*-algebra.

Corollary

If a C*-algebra is Cu-nuclear then it is nuclear.

LLP Local Lifting Property and exactness

Definition

A has Local Lifting Property (LLP) if for any C*-algebra C, any closed ideal I and $\forall u : A \to C/I$ u.c.p. (unital completely positive) is locally liftable: i.e. $\forall E \subset A$ f.d. oper. syst. $u_E : E \to C/I$ admits a lifting $u^E : E \to C$

LLP Local Lifting Property and exactness

Definition

A has Local Lifting Property (LLP) if for any C*-algebra C, any closed ideal I and $\forall u : A \to C/I$ u.c.p. (unital completely positive) is locally liftable: i.e. $\forall E \subset A$ f.d. oper. syst. $u_E : E \to C/I$ admits a lifting $u^E : E \to C$

Theorem (Kirschberg)

A has LLP if and only if $A \otimes_{min} B(H) = A \otimes_{max} B(H)$

LLP Local Lifting Property and exactness

Definition

A has Local Lifting Property (LLP) if for any C*-algebra C, any closed ideal I and $\forall u : A \to C/I$ u.c.p. (unital completely positive) is locally liftable: i.e. $\forall E \subset A$ f.d. oper. syst. $u_E : E \to C/I$ admits a lifting $u^E : E \to C$

Theorem (Kirschberg)

A has LLP if and only if $A \otimes_{min} B(H) = A \otimes_{max} B(H)$

Theorem

A C*-alg. is exact if

$$A \otimes_{min} (B(H)/K(H)) = (A \otimes_{min} B(H))/(A \otimes_{min} K(H))$$

constant Cuntz classes

Remark

Any two positive elements are homotopic in the cone of positive elements.

$$p(t) = ta + (1-t)b$$

constant Cuntz classes

Remark

Any two positive elements are homotopic in the cone of positive elements.

$$p(t) = ta + (1 - t)b$$

Remark

If $a \backsim b$ then is there a path $p(t)$ such that $p(t) \backsim a$?, i.e. constant rank

constant Cuntz classes

Remark

Any two positive elements are homotopic in the cone of positive elements.

$$p(t) = ta + (1-t)b$$

Remark

If $a \backsim b$ then is there a path $p(t)$ such that $p(t) \backsim a$?, i.e. constant rank

Theorem (A. Toms)

If a C*-algebra is simple separable exact $\mathcal{Z}$-stable approximate divisible and of real rank zero then: if $a \backsim b$ then a and b are connected by a path consisting of positive elements equivalent to a.

constant Cuntz classes

Theorem

Let A be a simple separable AI-algebras: if $a \backsim b$ then a and b are connected by a path consisting of positive elements equivalent to a.

constant Cuntz classes

Theorem

Let A be a simple separable AI-algebras: if $a \backsim b$ then a and b are connected by a path consisting of positive elements equivalent to a.

Remark

AI-algebras are not necessarily of real rank zero.

constant Cuntz classes

Theorem

Let A be a simple separable AI-algebras: if $a \backsim b$ then a and b are connected by a path consisting of positive elements equivalent to a.

Remark

AI-algebras are not necessarily of real rank zero.

Remark

Simple AI-algebras are $\mathcal{Z}$-stable. Hence it has strict comparison of positive elements.

constant Cuntz classes

Definition

A unital C*-algebra A is strongly K_1- surjective if the canonical map

$$\mathcal{U}(B + \mathbb{C}(1_A)) \longrightarrow K_1(A)$$

is surjective for every full hereditary subalgebra B of A.

constant Cuntz classes

Definition

A unital C*-algebra A is strongly K_1- surjective if the canonical map

$$\mathcal{U}(B + \mathbb{C}(1_A)) \longrightarrow K_1(A)$$

is surjective for every full hereditary subalgebra B of A.

Remark

Simple AI-algebras are K_1- surjective since their K_1 group is trivial.

constant Cuntz classes

Definition

A unital C*-algebra A is strongly K_1- surjective if the canonical map

$$\mathcal{U}(B + \mathbb{C}(1_A)) \longrightarrow K_1(A)$$

is surjective for every full hereditary subalgebra B of A.

Remark

Simple AI-algebras are K_1- surjective since their K_1 group is trivial.

Remark

A decomposition of non-compact elements is useful. Clear if we have Real rank zero property (A. Toms).

real rank zero property

Remark

If $< a > \in Cu(A)$ not compact (i.e. 0 is an accumulation point) then RR0 implies

$$< a >= \sup_i < q_i >, \quad < q_i > \; << \; < q_{i+1} >$$

q_i projections.

real rank zero property

Remark

If $< a > \in Cu(A)$ not compact (i.e. 0 is an accumulation point) then RR0 implies

$$< a >= \sup_i < q_i >, \quad < q_i > \ << \ < q_{i+1} >$$

q_i projections.

Remark

Any $a \in M_n(C(X))_+$ can be approximated by well supported.

well supported elements

Definition

Let X be a compact Hausdorff and $a \in M_n(C(X))_+$ with rank function lsc $f : X \to \mathbb{N} \cup \infty$ taking values $n_1 < \cdots < n_k$ so that
$F_i = \{x \in X, f(x) = n_i\}$.
a is **well supported** if there exists proj. $p_i \in M_n(C(\overline{F_i}))$, $i \in \{1, \ldots k\}$ such that

$$\lim_{r \to \infty} a^{\frac{1}{r}}(x) = p_i(x), \ \ x \in F_i$$

and $p_i(x) \leq p_j(x)$ for $i < j$ and $x \in F_i \cap F_j$

decomposition

Lemma

A simple AI-alg. and $a \in A_+$, $p_{n_k} \precsim a$. Then there exists a projection $p_1 \backsim p_{n_k}$ and positive element b_1 in $\overline{aAa}$ such that $b_1 p_1 = p_1 b_1 = 0$ and

$$d_\tau(a) = d_\tau(b_1) + d_\tau(p_1)$$

decomposition

Lemma

A simple AI-alg. and $a \in A_+$, $p_{n_k} \precsim a$. Then there exists a projection $p_1 \backsim p_{n_k}$ and positive element b_1 in $\overline{aAa}$ such that $b_1 p_1 = p_1 b_1 = 0$ and

$$d_\tau(a) = d_\tau(b_1) + d_\tau(p_1)$$

Remark

Repeat the previous Lemma to get a sequence of projections p_i. Then $a \backsim \sum \frac{1}{2^i} p_i$

If X is $[0,1]$ can assume positive elements are trivial

If X is $[0, 1]$ can assume positive elements are trivial

Remark

a positive is trivial if it corresponds to "trivial bundle", i.e.
there are mutually orthogonal projections p_i (correspond to trivial bundle) and cont. funct. g_i:

$$a \backsim \bigoplus g_i p_i$$

If X is $[0,1]$ can assume positive elements are trivial

Remark

a positive is trivial if it corresponds to "trivial bundle", i.e.
there are mutually orthogonal projections p_i (correspond to trivial bundle) and cont. funct. g_i:

$$a \backsim \bigoplus g_i p_i$$

Prop (A. Toms)

If X is compact Hausdorff, $a, b \in M_n(C(X))_+$ trivial then $a \precsim b$ iff $rank(a)(x) \leq rank(b)(x)$

Thank you

Thank you