Jet holonomies along a singular hypersurface

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Singular foliations

Definition (Androulidakis-Skandalis)

A (possibly singular) **foliation** $\mathcal{F}$ of a smooth manifold M is a $C^\infty(M)$-module of vector fields on M that is locally finitely-generated and closed under Lie bracket.

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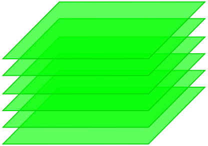

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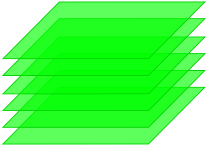
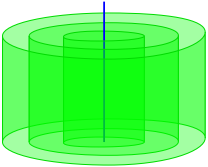


Frobenius: If $\mathcal{F}$ has "constant rank", M decomposes into leaves of constant dimension.

Stefan-Sussmann: If not, M still decomposes into leaves of different dimensions.



The C*-algebra of a foliation

Given a smooth, regular foliation $\mathcal{F}$ of a manifold M, there is an associated **foliation C*-algebra** $C^*(\mathcal{F})$. This is classical construction of Connes.

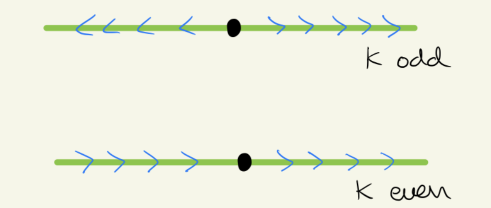
The C*-algebra is constructed using the **holonomy groupoid** or **graph** of $\mathcal{F}$. This is a Lie groupoid $G(\mathcal{F})$ over M introduced by Winkelnkemper.

$$\mathcal{F} \rightsquigarrow G(\mathcal{F}) \rightsquigarrow C^*(\mathcal{F})$$

Many authors have done work to extend these constructions to cases where $\mathcal{F}$ is a singular foliation (Androulidakis-Skandalis, Debord, Pradines-Bigonnet...).

Simplest examples

For each positive integer k, let $\mathcal{F}_k$ be the singular foliation of $\mathbb{R}$ singly-generated by $y^k \frac{d}{dy}$.

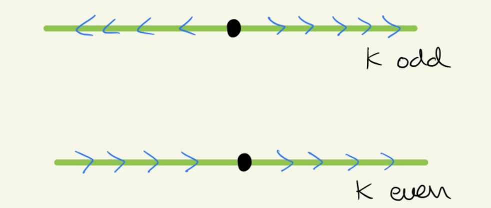


- For all k, the leaves are: $\mathbb{R}_-$, $\mathbb{R}_+$ and $\{0\}$.
- $G(\mathcal{F}_k) \cong \mathbb{R} \rtimes_{\phi^k} \mathbb{R}$ (transformation groupoid)
- $C^*(\mathcal{F}_k) \cong C_0(\mathbb{R}) \rtimes_{\phi^k} \mathbb{R}$.

Simplest examples

The isomorphism type of the C*-algebras only depends on the parity of k and they naturally sit in extensions:

$$0 \longrightarrow \mathbb{K} \oplus \mathbb{K} \longrightarrow C_0(\mathbb{R}) \rtimes_{\phi^k} \mathbb{R} \longrightarrow C^*(\mathbb{R}) \longrightarrow 0$$



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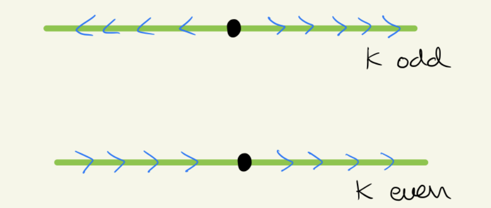

More interestingly, one has:

Theorem (F, 2020)

The smooth algebras $C_c^\infty(\mathbb{R} \rtimes_{\phi^k} \mathbb{R})$, $k = 1, 2, 3, \ldots$ are not isomorphic to one another.

Codimension-1 generalization

Fix a positive integer k.

We look at foliations which, locally, are the product of $\mathcal{F}_k$ and a trivial one leaf foliations. In a bit more detail:

Definition (Just for brevity...)

A foliation $\mathcal{F}$ of a connected manifold M is a k-**hypersurface foliation** if:

1. the leaves of $\mathcal{F}$ consist of a hypersurface L and the (open) components of its complement,
2. locally, $\mathcal{F}$ looks like $\mathbb{R}^n \times \mathbb{R}$ with the foliation generated by $\frac{d}{dx_1}, \ldots, \frac{d}{dx_n}, y^k \frac{d}{dy}$.

Examples of k-hypersurface foliations

Example

Let $M = \mathbb{R}^2$ and L be the x-axis. Consider:

- $\mathcal{F}_1$ generated by $y^2\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$
- $\mathcal{F}_2$ generated by $y^2\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

Then $\mathcal{F}_1 \neq \mathcal{F}_2$, but the diffeomorphism $\theta : M \to M$ given by $\theta(x, y) = (x, e^x y)$ has $\theta_*(\mathcal{F}_1) = \mathcal{F}_2$.

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Example

If we make x a periodic coordinate in the preceding example, so that $M = S^1 \times \mathbb{R}$, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are nonisomorphic (even though they have the same leaves and are locally isomorphic).

Groups of jets

Notation

Let J^k denote the group of kth order Taylor polynomials of diffeomorphisms of $\mathbb{R}$ fixing 0.

More concretely:

$$J^k = \{a_1 y + a_2 y^2 + \ldots a_k y^k : a_i \in \mathbb{R}, a_1 \neq 0\}$$

under the operation "compose and truncate".

- J^k is a solvable group.
- J^2 is isomorphic to the "ax+b group".
- There are canonical extensions $\mathbb{R} \to J^k \to J^{k-1}$, where the embedding of $\mathbb{R}$ is $t \mapsto y + ty^k$.

A jet blowup groupoid

Let Γ be an orientation-preserving subgroup of J^{k-1} and let $\Gamma_{\mathbb{R}} \subset J^k$ be the one-dimensional Lie group given as the preimage of Γ by the map $J^k \to J^{k-1}$.

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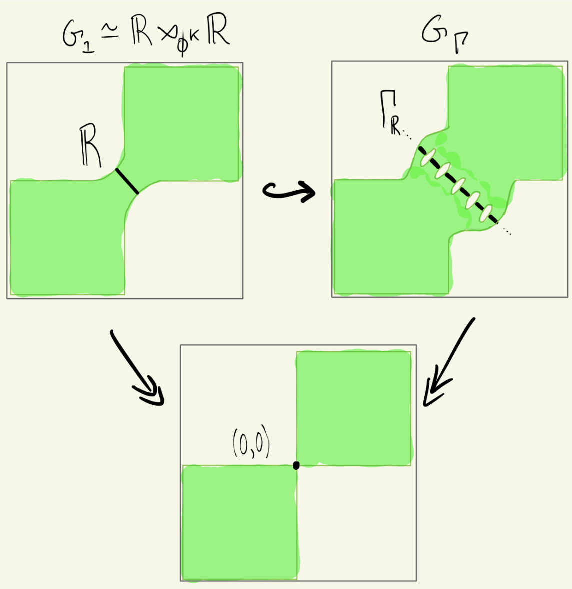
Theorem (F, 2021)

There is a smooth blowup G_Γ of the singular equivalence relation $\mathbb{R}^2_+ \cup \mathbb{R}^2_- \cup \{(0,0)\}$ replacing $(0,0)$ by $\Gamma_{\mathbb{R}}$.

$$G_\Gamma = \mathbb{R}^2_- \cup \mathbb{R}^2_+ \cup \Gamma_{\mathbb{R}}$$

If Γ is countable, G_Γ is a second-countable, Hausdorff Lie groupoid.

Note: $\Gamma_{\mathbb{R}}$ becomes the isotropy group of $0 \in \mathbb{R}$.



$\mathbb{R} \rtimes_{\phi^k} \mathbb{R}$ is always an open subgroupoid of G_Γ.

Classification by jet holonomies

Let $(M, \mathcal{F}, L)$ be a k-hypersurface foliation. Choose a transversal $T = \mathbb{R}$.

Theorem (F, 2021)

1. *$\mathcal{F}$ determines*[1] *a holonomy mapping $h_{\mathcal{F}} : \pi_1(L) \to J^{k-1}$.*
2. *Every homomorphism occurs as $h_{\mathcal{F}}$ for some $\mathcal{F}$.*
3. *$h_{\mathcal{F}}$ is a complete invariant of $\mathcal{F}$.*[2]

[1] Up to inner automorphisms of J^{k-1}.

[2] Restricted to a tubular neighbourhood of L.

The holonomy groupoid

Assume now $(M, \mathcal{F}, L)$ is a transversely-oriented k-hypersurface foliation (we may take $M = L \times \mathbb{R}$).

Theorem (F, 2021)

The holonomy groupoid $G(\mathcal{F})$ of a k-hypersurface foliation is a second-countable, Hausdorff Lie groupoid.

Moreover, picking a transversal $T = \mathbb{R}$, the restriction $G(\mathcal{F})_T$ is isomorphic to the blowup groupoid G_Γ discussed before, where $\Gamma \subset J^{k-1}$ is the range of the holonomy map $\pi_1(L) \to J^{k-1}$.

In particular, $C^*(\mathcal{F}) \sim_{Morita} C^*(G_\Gamma)$.

Fundamental K-theory class

A. Connes, *Noncommutative Geometry*

pp. 255

Theorem 6. *Let Γ be a countable group of orientation-preserving homeomorphisms of $\mathbb{S}^1$ and $A = C(\mathbb{S}^1) \rtimes_r \Gamma$ be the reduced crossed product C^*-algebra. Then the canonical homomorphism $i : C(\mathbb{S}^1) \to A$ is an injection of $K_1(C(\mathbb{S}^1)) = \mathbb{Z}$ in $K_1(A)$.*

pp.258

Theorem 9. *Let (V, F) be a transversely oriented foliation of codimension 1; then $[V/F]^*$ is a non-torsion element of $K_1(C_r^*(V, F))$.*

Fundamental K-theory class

If $(M, \mathcal{F}, L)$ is a transversely-oriented, k-hypersurface foliation, we may make the following definition (independent of choice of positively-oriented transversal T).

Definition

The **fundamental class** $[M/\mathcal{F}]^* \in K_0(C^*(\mathcal{F}))$ is the image of the positively-oriented generator of $K_1(C_0(T))$ under:

$$K_1(C_0(T)) \overset{\cong}{\to} K_0(C_0(T) \rtimes_{\phi^k} \mathbb{R}) \to K_0(C^*(G_\Gamma)) \overset{\cong}{\to} K_0(C^*(\mathcal{F}))$$

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Question

Is the fundamental class of a transversely-oriented k-hypersurface foliation always non torsion?

I don't know the answer right now!

Boils down to the following:

Question

Given a group $\Gamma \subset J^{k-1}$, is the natural map

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- One approach: try to extend a 2-trace on $C_0(\mathbb{R}) \rtimes_{\phi^k} \mathbb{R}$ (Elliot-Natsume-Nest) to $C^*(G_\Gamma)$. Unfortunately, ϕ^k has no invariant measure or we could use a trace instead. This would be in the spirit of what Connes did.

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- Alternatively, can reformulate as a question about the range of index map $K_1(C^*(\mathbb{R})) \to \mathbb{Z} \oplus \mathbb{Z}$ coming from $0 \to \mathbb{K} \oplus \mathbb{K} \to C^*(G_\Gamma) \to C^*(\Gamma_\mathbb{R}) \to 0$.

Thanks for listening!

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- M. Francis: A Dixmier-Malliavin theorem for Lie groupoids. *arXiv:2009.13760* (2020). To appear J. Lie theory.
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- M. Francis: Groupoids and algebras of certain singular foliations with finitely many leaves. (PhD thesis).

Theorem

Let $\Gamma \subset J^{k-1}$ *be any orientation preserving group of jets. Then* $K_0(C_0(\mathbb{R}) \rtimes_{\phi^k} \mathbb{R})$ *is injective if and only if*

$$\mathrm{ind} : K_1(C^*(\Gamma_{\mathbb{R}})) \to \mathbb{Z} \oplus \mathbb{Z}$$

arising from the extension

$$0 \longrightarrow \mathbb{K} \oplus \mathbb{K} \longrightarrow C^*(G_\Gamma) \longrightarrow C^*(\Gamma_{\mathbb{R}}) \longrightarrow 0$$

has range $\mathbb{Z} \cdot (1, 1)$ *if* k *is odd, resp.* $\mathbb{Z} \cdot (1, -1)$ *if* k *is even.*

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{K} \oplus \mathbb{K} \arrow[r] \arrow[d, equal] & C_0(\mathbb{R}) \rtimes_{\phi^k} \mathbb{R} \arrow[r] \arrow[d] & C^*(\mathbb{R}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{K} \oplus \mathbb{K} \arrow[r] & C^*(G_\Gamma) \arrow[r] & C^*(\Gamma_{\mathbb{R}}) \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & K_1(C_0(\mathbb{R})) \arrow[d, "\cong"] \\
K_1(C^*(\mathbb{R})) \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d, equal] & K_0(C_0(\mathbb{R}) \rtimes_{\phi^k} \mathbb{R}) \arrow[d] \\
K_1(C^*(\Gamma_{\mathbb{R}}) \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & K_0(C^*(G_\Gamma))
\end{tikzcd}
$$

$$\begin{aligned}\Gamma &= \{x + mx^4 + nx^5 : m, n \in \mathbb{Z}\} \\ &\cong \mathbb{Z} \times \mathbb{Z}\end{aligned}$$

$$\begin{aligned}\Gamma_{\mathbb{R}} &= \{x + mx^4 + nx^5 + tx^6 : m, n \in \mathbb{Z}; t \in \mathbb{R}\} \\ &\cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}\end{aligned}$$

$$C^*(\Gamma_{\mathbb{R}}) \cong C(\mathbb{T}^2) \otimes C_0(\mathbb{R})$$

$$K_1(C^*(\Gamma_{\mathbb{R}})) \cong K^0(\mathbb{T}^2)$$