The Cuntz semigroup of continuous functions into a strongly self-absorbing C*-algebra

Aaron Tikuisis
aptikuis@math.toronto.edu

University of Toronto

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The Cuntz semigroup

Ordered semigroup, constructed as follows.

- For $a, b \in (A \otimes \mathcal{K})_+$, $a \precsim_{Cuntz} b$ if
$$a = \lim s_n^* b s_n,$$
for some $(s_n) \subset A \otimes \mathcal{K}$.
- $a \sim_{Cuntz} b$ if $a \precsim_{Cuntz} b$ and $b \precsim_{Cuntz} a$.
- $\mathcal{C}u(A) = (A \otimes \mathcal{K})_+ / \sim_{Cuntz}$
- $[a] + [b] := [a' + b']$ where $a \sim_{Cuntz} a', b \sim_{Cuntz} b'$ and $a' \perp b'$.

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For $f, g \in C(X)_+$:

- $f \precsim_{Cuntz} g$ if and only if

$$\{x \in X : f(x) > 0\} \subseteq \{x \in X : g(x) > 0\}.$$

However, Cuntz comparison for $C(X) \otimes \mathcal{K}$ isn't so simple.

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- If $A \otimes \mathcal{K}$ is finite then $p \sim_{Cuntz} q$ if and only if

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- The Cuntz semigroup hasn't been computed for many C*-algebras, apart from those that are both simple and $\mathcal{Z}$-stable algebras.
- Coming up: computations of the Cuntz semigroup of $C(X, \mathcal{D})$ for certain algebras $\mathcal{D}$ (they are strongly self-absorbing).
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General considerations for $C(X, A)$

Let $a, b \in C(X, A)_+$

- $a \precsim_{Cuntz} b$ implies that $a(x) \precsim_{Cuntz} b(x)$ for all x.
- Converse does not hold: a, b could be inequivalent projections in $C(X, \mathcal{K})$ with the same rank.
- More generally, if $K \subseteq X$ is closed, $p \in A$ is a projection, and
$$a(x) \sim_{Cuntz} b(x) \sim_{Cuntz} p \text{ for all } x \in K,$$
then $a \precsim_{Cuntz} b$ implies that $[a|_K] \leq [b|_K]$.
In this situation, 0 is not an accumulation point of the spectrum of $a|_K$ (or $b|_K$), so this is the same as
$$\chi_{(0,\infty)}(a|_K) \sim_{M-vN} \chi_{(0,\infty)}(b|_K).$$

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$$\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$$

via an isomorphism ϕ which is approx. unitarily equivalent to

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Known examples

$$
\begin{tikzcd}[column sep=small]
M_{n^\infty} \arrow[dr, dash] & \cdots & M_{m^\infty} \arrow[dl, dash] \\
& M_{(mn)^\infty} \arrow[d, dash] & \\
& \vdots \arrow[d, dash] & \\
& \mathcal{Q} &
\end{tikzcd}
$$

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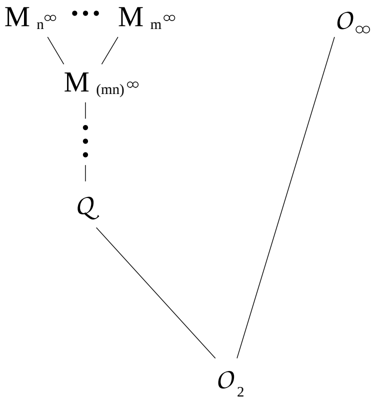

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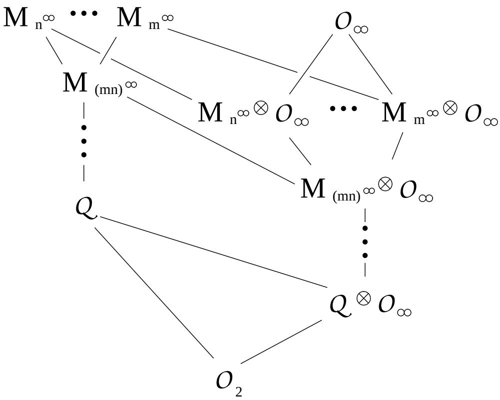

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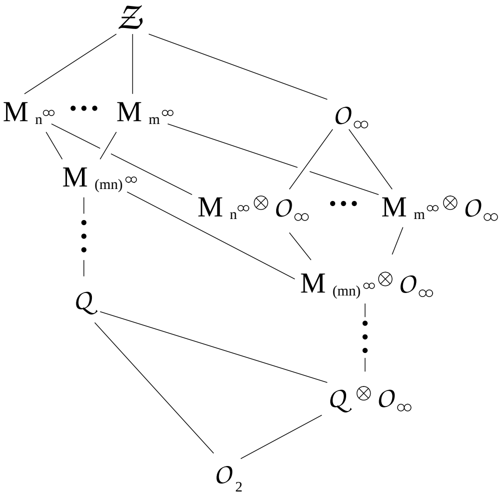

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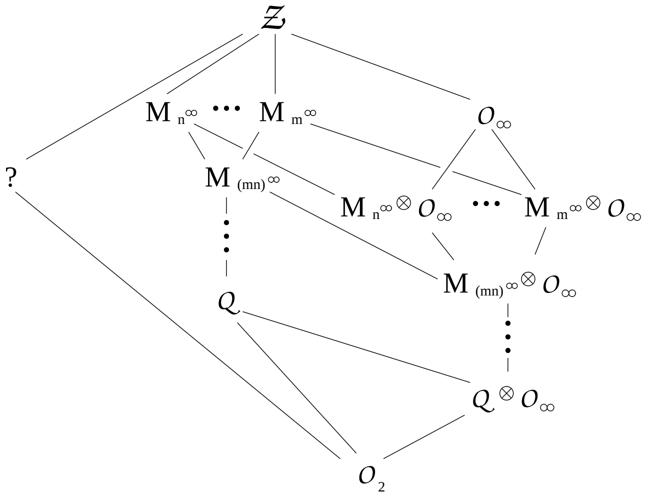

Simple $\mathcal{Z}$-stable C*-algebras

Theorem (Brown-Perera-Toms, Elliott-Robert-Santiago)

For A simple, exact, finite and $\mathcal{Z}$-stable, then

$$\mathcal{C}u(A) = (V(A)\backslash\{0\}) \amalg \mathrm{Lsc}(T(A), [0,\infty]).$$

In particular, $\mathcal{C}u(\mathcal{Z}) = \{1, 2, 3, \ldots\} \amalg [0, \infty]$.

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$C(X, \mathcal{D})$, $\mathcal{D}$ S.S.A. and purely infinite

X compact, Hausdorff, metrizable (ie. $C(X)$ separable)
By Kirchberg-Rordam ('00), $C(X, \mathcal{D})$ is purely infinite, so for $a, b \in C(X, \mathcal{D} \otimes \mathcal{K})_+$, $a \precsim_{Cuntz} b$ if and only if

$$\{x \in X : a(x) \neq 0\} \subseteq \{x \in X : b(x) \neq 0\}.$$

Moreover, for any open set $U \subset X$, there exists $[a] \in \mathcal{C}u(C(X, \mathcal{D}))$ such that

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ie. $\mathcal{C}u(C(X, \mathcal{D}))$ looks like the topology of X.

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Theorem. (T)

For $a, b \in C(X, \mathcal{D})_+$, $a \precsim_{Cuntz} b$ if and only if

(i) $a(x) \precsim_{Cuntz} b(x)$ for all x and

(ii) Whenever $K \subseteq X$ is closed, $p \in \mathcal{D} \otimes \mathcal{K}$ is a projection, and

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Condition (ii) is really about projections in $C(K,\mathcal{K})$, since

$$[a|_K] = [a']$$

for some $a' \in C(K,M_m) \subseteq \mathcal{D}$.

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Let $V_c(Y, \mathcal{D})$ be the set of equivalence classes of continuous projections $p : Y \to \mathcal{D} \otimes \mathcal{K}$ under the relation $p \sim q$ if $p|_K \sim q|_K$ for all compact $K \subseteq Y$.

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(ii) Whenever $K \subseteq X$ is closed, $p \in \mathcal{D} \otimes \mathcal{K}$ is a projection, and
$$a(x) \sim_{Cuntz} b(x) \sim_{Cuntz} p \text{ for all } x \in K,$$
then we have $[a|_K] = [b|_K]$ in $V(C(K, \mathcal{D}))$.

Let $V_c(Y, \mathcal{D})$ be the set of equivalence classes of continuous projections $p : Y \to \mathcal{D} \otimes \mathcal{K}$ under the relation $p \sim q$ if $p|_K \sim q|_K$ for all compact $K \subseteq Y$. Then (ii) says
$$[\chi_{(0,\infty)}(a|_Y)] = [\chi_{(0,\infty)}(b|_Y)] \text{ in } V_c(Y, \mathcal{D})$$
where $Y = \{x \in X : a(x) \sim_{Cuntz} b(x) \sim_{Cuntz} p\}$.

$C(X, \mathcal{D})$, $\mathcal{D}$ UHF of infinite type

X compact, Hausdorff, metrizable

Theorem. (T)

For any lower semicontinuous function $f : X \to \mathcal{C}u(\mathcal{D})$ and family $([a_{[p]}])_{[p] \in V(\mathcal{D})}$, where

$$[a_{[p]}] \in V_c(\{x \in X : f(x) = [p]\}, \mathcal{D})), \ [a_{[p]}(x)] = [p] \ \forall x,$$

there exists $[a] \in \mathcal{C}u(C(X, \mathcal{D}))$ such that

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The Cuntz semigroup of $C(X, \mathcal{D})$ consists of $(f, ([a_{[p]}])_{[p] \in V(\mathcal{D})})$ where

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In fact, this description applies to any simple AF algebra A except M_n or $\mathcal{K}$.

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$C(X, \mathcal{Z})$

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Description of $\mathcal{C}u(C(X, \mathcal{Z}))$ (T)

The Cuntz semigroup of $C(X, \mathcal{Z})$ consists of $(f, ([p_k])_{k=1}^{\infty})$ where

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We have $(f, ([p_k])_{k=1}^{\infty}) \leq (g, ([q_k])_{k=1}^{\infty})$ if and only if

(i) $f(x) \leq g(x)$ for all x, and

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