The Cuntz Semigroup for Commutative C^*-algebras

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Joint work with Leonel Robert

Constructing the Cuntz Semigroup

Two Constructions of Cu(A):

- With elements of $\bigcup_n \mathfrak{M}_n \otimes A$
  - $a \precsim_{Cuntz} b$ if $a = \lim s_n b t_n$, some $(s_n), (t_n)$
  - $[a] + [b] = [a \oplus b]$

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Two Constructions of Cu(A):

- With finitely generated Hilbert modules
  - $\precsim_{Cuntz}$ described in terms of $\subset$, $\cong$
  - $\precsim_{Cuntz}$ weaker than $\cong\subset$
  - $[H] + [K] = [H \oplus K]$

The correspondence is: $a \in \mathfrak{M}_n \otimes A \quad \mapsto \overline{a^* A^n}$.

$\overline{a^* A^n} \cong \overline{b^* A^n}$ iff for some b', $\overline{b^* A^n} = \overline{b'^* A^n}$ and $|a| \sim_{M-vN} |b|$ (ie. $\exists x, |a| = x^* x, xx^* = |b'|$)

- Let $\mathrm{Cu}_s(A) = \mathrm{Cu}(\mathcal{K} \otimes A)$.

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The Cuntz Semigroup

- Cuntz semigroup promises to be a useful tool in the Classification Program
- Computations of $\mathrm{Cu}(A)$ are rare:
  - Brown-Perera-Toms: A simple, unital, $\mathcal{Z}$-stable, stably finite
  - $A = C(X)$, $\dim X \leq 1$ or $\dim X = 2$ and $H^2(X) = 0$.

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Using Open Projections

Definition.

An **open projection** is a projection in A^{**} which is an increasing limit of elements of A.

- Assume A is sep., so all open projections are $\chi_{(0,\infty)}(a)$, $a \in A_+$.
- Atomic representation is faithful for open projections, so view open projections in this representation.

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Using Open Projections

Use open projections to represent Cuntz elements

- For $a, b \in \mathfrak{M}_n \otimes A$, $\chi_{(0,\infty)}(|a|) = \chi_{(0,\infty)}(|b|)$ iff $\overline{a^* A^n} = \overline{b^* A^n}$.
- $\overline{a^* A^n} \cong \overline{b^* A^n}$ iff $\chi_{(0,\infty)}(|a|) \sim_{M-vN} \chi_{(0,\infty)}(|b|)$ where the partial isometry occurs in the polar decomposition of an element of $\mathfrak{M}_n \otimes A$.
- Hilbert module assoc. to open projection p is a submodule of Hilbert module assoc. to q iff $p \leq q$.
- No simple formulation of Cuntz order relation for open projections.

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Open Projections in $C_0(X)$

X 2nd ctble. l.c. Hausdorff

Open projections are particularly useful for studying $\mathrm{Cu}(C_0(X))$.

- Atomic representation of $C_0(X) \otimes \mathfrak{M}_n$ gives $L^\infty(X) \otimes \mathfrak{M}_n$, so functional calculus is done pointwise
- An open projection p for $\mathfrak{M}_n \otimes C_0(X)$ is given by a compatible family $(p_i)_{i=0}^n$ of continuous projections:
  - p_i defined on open set U_i, where $U_0, \ldots, U_n$ cover X
  - p_i has rank i
  - $p_i \leq p_j$ on $U_i \cap U_j$ for $i \leq j$
  - p is given by $\bigvee p_i$ (ie. $p(x) = p_i(x)$ for greatest i s.t. $p_i(x)$ is defined).

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- A partial isometry from a polar decomposition is given by a compatible family $(v_i)_{i=0}^n$ of continuous partial isometries:
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Open Projections in $C_0(X)$

X 2nd countable, l.c., Hausdorff

Consider the case that p, q are constant rank open projections (thus belong to $C_b(X)$):

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- **Not** the same as $p \precsim_{M-vN} q$ in $C_b(X)$.

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Dimension Three Spectrum

X be 2nd countable, l.c., Hausdorff, $\dim X \leq 3$.

For open projection p, let $R_{=i}(p) := \{x \in X : \mathrm{Rank}\ p(x) = i\}$.

Theorem 1. (Robert-T)

For open projections p, q of $\mathcal{K} \otimes C_0(X)$,
$p \precsim_{Cuntz} q$ iff for each $i, j \in \mathbb{N}$,

$$p|_{R_{=i}(p) \cap R_{=j}(q)} \precsim_{Cuntz} q|_{R_{=i}(p) \cap R_{=j}(q)}.$$

Theorem 2. (Robert-T)

Given any bounded l.s.c. $r : X \to \mathbb{N}$ and any (not necessarily compatible) family of continuous projections (p_i) s.t.

- p_i is defined on $r^{-1}(\{i\})$
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$\exists$ open projection p of $\mathfrak{M}_m \otimes C_0(X)$ s.t. $p|_{r^{-1}(\{i\})} \sim_{Cuntz} p_i$.

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Description of $\mathrm{Cu}(C_0(X))$.

Elements of $\mathrm{Cu}(C_0(X))$ are pairs $(r, (p_i))$ where

- $r : X \to \mathbb{N}$ is bounded and l.s.c.
- For each $i \in \mathbb{N}$, p_i is a constant rank i element of $\mathrm{Cu}(C_0(r^{-1}(\{i\})))$

$(r, (p_i)) \leq (r', (p_i'))$ iff $p_i \precsim_{Cuntz} p_j$ where both defined, $\forall i, j$

- Have Cuntz order of constant rank elements in terms of M-vN equivalence of restrictions, so this describes $\mathrm{Cu}(C_0(X))$ in terms of $V(Y)$ for compact $Y \subset X$.
- For $\mathrm{Cu}_s(C_0(X))$, Thm. 1 describes the order, but don't have a nice description of what data arises.

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X be 2nd countable, l.c., Hausdorff, $\dim X \leq 3$.

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Outlook

Applications of dim. 3 description?

- Can show $\mathrm{Cu}_s(C_0(X))$ has **weak cancellation**, ie. if $a + c << b + c$ then $a \leq b$.
- May be interesting to look at $\mathrm{Cu}_s(A)$ where A is AH with dimension bounded by 3
  - Gong: Includes all simple AH with slow dimension growth
  - Could help understand Elliott-Gong-Li classification
  - Includes $B \otimes C_0((0,1])$ where B is AH with dimension bounded by 2
  - Robert-Santiago: $\mathrm{Cu}_s(B \otimes C_0((0,1]))$ occurs in classifying homomorphisms $C_0((0,1]) \to B$, for such B.

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More General Commutative C*-algebras

- General goal is to describe $\mathrm{Cu}(C_0(X))$ for $\dim X < \infty$, in terms of $V(Y)$, $Y \subset X$ closed.

Conjecture.

For $[a], [b] \in \mathrm{Cu}(C_0(X))$, $[a] \leq [b]$ iff for every $Y \subset X$ compact, and every continuous projection p on Y,

$$p \precsim_{Cuntz} a|_Y \Rightarrow p \precsim_{Cuntz} b|_Y.$$

- Thm. 1 $\Rightarrow$ conjecture holds for $\dim X \leq 3$.
- Also, conjecture holds when $[a], [b]$ have constant rank.
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