Operator-valued functions that are integrable against a positive, operator-valued measure

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Joint work with Christopher Ramsey (MacEwan University) and earlier work with Doug Farenick (University of Regina) and Darian McLaren (University of Waterloo)

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The Setting

- X is a locally compact Hausdorff space
- $\mathcal{O}(X)$ is the σ-algebra of Borel sets of X
- $\mathcal{H}$ is a finite or separable Hilbert space
- $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$
- $\mathcal{T}(\mathcal{H})$ is the Banach space of all trace-class operators: all operators in $\mathcal{B}(\mathcal{H})$ which have a finite trace under any orthonormal basis
- The convex subset $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ of all positive, trace-one trace-class operators ρ (called *states* or density operators)

We are interested in positive operator-valued measures $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ and ν-integrable functions $X \to \mathcal{B}(\mathcal{H})$. **Why?** The desire for a notion of an operator-valued averaging, i.e., the quantum expected value of a quantum random variable. To define majorization through the use of bistochastic operators in this setting.

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Positive Operator-valued Measures

Definition

A map $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})_+$ is a *positive operator-valued measure (POVM)* if it is ultraweakly countably additive: for every countable collection $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{O}(X)$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k)\,,$$

where the convergence on the right side of the equation above is with respect to the ultraweak topology of $\mathcal{B}(\mathcal{H})$, that is,

$$\mathrm{Tr}\left(s\sum_{k=1}^{n} \nu(E_k)\right) \to \mathrm{Tr}\left(s\sum_{k=1}^{\infty} \nu(E_k)\right), \quad \forall s \in \mathcal{S}(\mathcal{H}).$$

Absolute Continuity

Definition

A (classical or operator-valued) measure ω_1 is *absolutely continuous* with respect to either a classical or operator-valued measure ω_2, denoted $\omega_1 \ll_{\mathrm{ac}} \omega_2$, if $\omega_1(E) = 0$ whenever $\omega_2(E) = 0$, where $E \in \mathcal{O}(X)$ (for classical measures, $\mathcal{O}(X)$ is typically denoted by Σ) and 0 is interpreted as either the scalar zero or the zero operator, as applicable.

Let $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$. For a fixed state $\rho \in \mathcal{S}(\mathcal{H})$, the induced complex measure ν_ρ on X is defined by $\nu_\rho(E) = \mathrm{Tr}(\rho\nu(E))$ for all $E \in \mathcal{O}(X)$. Note: ν and ν_ρ are mutually absolutely continuous for any full-rank $\rho \in \mathcal{S}(\mathcal{H})$.

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Building a Radon-Nikodým derivative

Let $\nu_{i,j}$ be the complex measure defined by $\nu_{i,j}(E) = \langle \nu(E) e_j, e_i \rangle$, $E \in \mathcal{O}(X)$, where $\{e_k\}$ form an orthonormal basis for $\mathcal{H}$. Let $\rho \in \mathcal{S}(\mathcal{H})$ be full-rank. Then $\nu_{i,j} \ll_{\mathrm{ac}} \nu_\rho$ and so, by the classical Radon-Nikodým theorem, there is a unique $\frac{d\nu_{i,j}}{d\nu_\rho} \in L_1(X, \nu_\rho)$ such that

$$\nu_{i,j}(E) = \int_E \frac{d\nu_{i,j}}{d\nu_\rho} d\nu_\rho, \ E \in \mathcal{O}(X).$$

One can then define the *Radon-Nikodým derivative* of ν with respect to ν_ρ to be

$$\frac{d\nu}{d\nu_\rho} = \sum_{i,j \geq 1} \frac{d\nu_{i,j}}{d\nu_\rho} \otimes e_{i,j}.$$

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Quantum Random Variables

Definition

An operator-valued function $f : X \to \mathcal{B}(\mathcal{H})$ that is Borel measurable (that is, the associated complex-valued functions $x \to \mathrm{Tr}(sf(x))$ are Borel measurable functions for every state $s \in \mathcal{S}(\mathcal{H})$) is called a *quantum random variable*.

The Radon-Nikodým derivative $\frac{d\nu}{d\nu_\rho}$ is said to exist if it is a quantum random variable; i.e. it takes every x to a bounded operator. If $\frac{d\nu}{d\nu_{\rho_0}}$ exists for some full-rank $\rho_0 \in \mathcal{S}(\mathcal{H})$, then $\frac{d\nu}{d\nu_\rho}$ exists for all full-rank $\rho \in \mathcal{S}(\mathcal{H})$, so there is no need to specify a particular full-rank ρ_0.

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Integrability of a Quantum Random Variable wrt a POVM

Definition

Let $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM such that $\frac{d\nu}{d\nu_\rho}$ exists. A positive quantum random variable $f : X \to \mathcal{B}(\mathcal{H})$ is *ν-integrable* if the function

$$f_s(x) = \mathrm{Tr}\left(s \left(\frac{d\nu}{d\nu_\rho}(x) \right)^{1/2} f(x) \left(\frac{d\nu}{d\nu_\rho}(x) \right)^{1/2} \right)$$

is ν_ρ-integrable for every state $s \in \mathcal{S}(\mathcal{H})$.

If f is ν-integrable then the integral of f with respect to ν, denoted $\int_X f d\nu$, is implicitly defined by the formula

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Notes

A particularly nice case: If $\nu = \mu I_{\mathcal{H}}$ for a positive complex measure μ then we know that $\frac{d\nu}{d\nu_\rho} = I_{\mathcal{H}}$ and if $f = [f_{i,j}]$ is taken with respect to an orthonormal basis in $\mathcal{H}$ then integration is defined entrywise:

$$\int_X f d\nu = \left[\int_X f_{i,j} d\mu \right].$$

What about Quantum Random Variables that are **not** Positive?
Any quantum random variable $f : X \to \mathcal{B}(\mathcal{H})$ can be decomposed as the sum of four positive quantum random variables (e.g. $(\mathrm{Re} f)_+, (\mathrm{Re} f)_-, (\mathrm{Im} f)_+$, and $(\mathrm{Im} f)_-$). The definition of ν-integrable can thus be extended to arbitrary quantum random variables provided all four positive functions are ν-integrable.

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A generalization of the L^1-norm in the POVM context

Definition

Let $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ and define

$$\mathcal{L}^1_{\mathcal{H}}(X,\nu) = \mathrm{span}\{f : X \to \mathcal{B}(\mathcal{H}) : \nu\text{-integrable, positive quantum random variable}\}.$$

For every $f \in \mathcal{L}^1_{\mathcal{H}}(X,\nu)$ define

$$\|f\|_1 = \inf\left\{ \left\| \int_X \sum_{k=1}^{4} f_k \, d\nu \right\| : f = f_1 - f_2 + i(f_3 - f_4), f_k \in \mathcal{L}, f_k \geq 0, k = 1, \ldots, 4 \right\}.$$

We may write $\|f\|_{1,\nu}$ to emphasize the POVM ν that f is being integrated against.

This is a semi-norm on $\mathcal{L}^1_{\mathcal{H}}(X,\nu)$.

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The von Neumann algebra of essentially bounded quantum random variables

Let

$$L^\infty_{\mathcal{H}}(X,\nu) = \{h : X \to \mathcal{B}(\mathcal{H}) \text{ qrv } : \exists M \geq 0, \|h(x)\| \leq M \text{ a.e wrt } \nu\}$$
$$= L^\infty(X,\nu_\rho) \bar{\otimes} \mathcal{B}(\mathcal{H})$$

Note that the norm this comes with is defined as

$$\|f(x)\|_\infty := \Big\| \|f(x)\| \Big\|_{L^\infty(X,\nu_\rho)}$$

since $\|f(x)\| \in L^\infty(X,\nu_\rho)$.

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Proposition

Suppose $\mathcal{H} = \mathbb{C}^n$, $\nu \in \mathrm{POVM}_{\mathcal{H}}(X)$ *such that* $\frac{d\nu}{d\nu_\rho} \in M_n$ *is invertible almost everywhere* ($\frac{d\nu}{d\nu_\rho} \in M_n^{-1}$ *a.e.), and* $\frac{d\nu}{d\nu_\rho}, \frac{d\nu}{d\nu_\rho}^{-1} \in L^\infty_{\mathcal{H}}(X, \nu)$. *For* $f \in \mathcal{L}^1_{\mathcal{H}}(X, \nu)$ *self-adjoint we have*

$$\|f\|_1 \leq \left\| \int_X |f(x)| d\nu \right\| \leq \left\| \int_X \|f(x)\| I_n d\nu \right\| \leq n \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \left\| \frac{d\nu}{d\nu_\rho}^{-1} \right\|_\infty \|f\|_1.$$

Recall for $\nu \in \mathsf{POVM}_{\mathcal{H}}(X)$ we have

$$\mathcal{L}^1_{\mathcal{H}}(X,\nu) = \mathrm{span}\{f : X \to \mathcal{B}(\mathcal{H}) : \nu\text{-integrable, positive quantum random variable}\}.$$

Define $\mathcal{I} = \{f \in \mathcal{L}^1_{\mathcal{H}}(X,\nu) : \|f\|_1 = 0\}$ and let $L^1_{\mathcal{H}}(X,\nu) = \mathcal{L}^1_{\mathcal{H}}(X,\nu)/\mathcal{I}$. The previous lemma implies that the 1-topology on $L^1_{\mathcal{H}}(X,\nu)$ is stronger than the topology $(f_n)_s \to f_s$ for all $s \in \mathcal{S}(\mathcal{H})$.

Theorem

$L^1_{\mathcal{H}}(X,\nu)$ is a Banach space, that is, it is complete in the 1-norm for $\nu \in \mathsf{POVM}_{\mathcal{H}}(X)$ where $\frac{d\nu}{d\nu_\rho}$ exists.

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How to Relate $L^\infty_{\mathcal{H}}(X,\nu)$ and $L^1_{\mathcal{H}}(X,\nu)$

Proposition

Suppose $\frac{d\nu}{d\nu_\rho}(x) \in \mathcal{B}(\mathcal{H})^{-1}$ *for all* $x \in X$ *and* $\frac{d\nu}{d\nu_\rho}, \frac{d\nu}{d\nu_\rho}^{-1} \in L^\infty_{\mathcal{H}}(X,\nu)$. *There is a natural inclusion of* $L^\infty_{\mathcal{H}}(X,\nu)$ *in* $L^1_{\mathcal{H}}(X,\nu)$ *with*

$$\|g\|_1 \leq 2\|g\|_\infty \|\nu(X)\|, \quad \forall g \in L^\infty_{\mathcal{H}}(X,\nu).$$

Moreover, $L^\infty_{\mathcal{H}}(X,\nu)$ *is dense in* $L^1_{\mathcal{H}}(X,\nu)$ *in the state topology,* $(f_n)_s \to f_s$ *for all* $s \in \mathcal{S}(\mathcal{H})$.

Finite vs Infinite Dimensions

This proposition implies that if $\mathcal{H} = \mathbb{C}^n$ then $L^1_{\mathcal{H}}(X, \nu) = \overline{L^\infty_{\mathcal{H}}(X, \nu)}^{\|\cdot\|_1}$. In infinite dimensions this will not be the case: consider $X = [0, 1]$, $\mathcal{H}$ countably infinite dimensional, and $\nu = \mu I_{\mathcal{H}}$ where μ is Lebesgue measure. Then $f(x) = \sum_{n \geq 1} 2^n \chi_{(\frac{1}{2^n}, \frac{1}{2^{n-1}})}(x) e_{n,n}$ cannot be approximated by essentially bounded functions in the 1-norm.

Decreasing Rearrangements

One can define continuous majorization in the context of functions in L^1:

Definition

Let $(X, \mathcal{O}(X), \mu)$ be a finite positive measure space and $f \in L^1(X, \mu)$. The *distribution function* of f is $d_f : \mathbb{R} \to [0, \mu(X)]$ defined by

$$d_f(s) = \mu(\{x : f(x) > s\})$$

and the *decreasing rearrangement* of f is $f^{\downarrow} : [0, \mu(X)] \to \mathbb{R}$ defined by

$$f^{\downarrow}(t) = \sup\{s : d_f(s) \geq t\}.$$

Majorization

Definition

Let $(X_i, \mathcal{O}(X_i), \mu_i)$, $i = 1, 2$, be finite measure spaces for which $a = \mu_1(X_1) = \mu_2(X_2)$. Then $f \in L^1(X_1, \mu_1)$ is *majorized* by $g \in L^1(X_2, \mu_2)$, denoted $f \prec g$, if

$$\int_0^t f^{\downarrow} dx \leq \int_0^t g^{\downarrow} dx \quad \forall\, 0 \leq t \leq a$$

$$\text{and } \int_0^a g^{\downarrow} dx = \int_0^a f^{\downarrow} dx,$$

where integration is against Lebesgue measure.

Bistochastic Operators

An operator $B : L^1(X_1, \mu_1) \to L^1(X_2, \mu_2)$ between finite measure space where $\mu_1(X_1) = \mu_2(X_2)$ is called *bistochastic, doubly stochastic*, or *Markov*, if

1. B is positive
2. $\displaystyle\int_{X_2} Bf d\mu_2 = \int_{X_1} f d\mu_1$, and
3. $\mathrm{B1} = 1$

where 1 here refers to the constant function 1 in each of the spaces $L^1(X_i, \mu_i), i = 1, 2$.

Combining results of Hardy-Littlewood-Pólya, Chong, Ryff, and Day

Theorem

Let $(X_i, \mathcal{O}(X_i), \mu_i)$, $i = 1, 2$, be finite measure spaces for which $\mu_1(X_1) = \mu_2(X_2)$. If $f \in L^1(X_1, \mu_1)$ and $g \in L^1(X_2, \mu_2)$ then the following are equivalent:

- $f \prec g$
- $\int_{X_1} \psi(f(x))dx \leq \int_{X_2} \psi(g(x))dx$ *for all convex functions* $\psi : \mathbb{R} \to \mathbb{R}$
- *There is a bistochastic operator B such that $Bg = f$.*

Bistochastic Operators

Definition

A linear operator B is called a *bistochastic operator* on $L^1_{\mathcal{H}}(X,\nu)$ if

1. B is positive,
2. $\int_X Bf d\nu = \int_X f d\nu, \quad \forall f \in L^1_{\mathcal{H}}(X,\nu)$,
3. $BI_{\mathcal{H}} = I_{\mathcal{H}}$,

where $I_{\mathcal{H}}$ above refers to the constant function $I_{\mathcal{H}}$ in $L^1_{\mathcal{H}}(X,\nu)$. The set of all bistochastic operators on $L^1_{\mathcal{H}}(X,\nu)$ is denoted by $\mathfrak{B}(X,\nu)$.

Proposition

Every bistochastic operator is contractive with respect to the $\|\cdot\|_1$-norm.

The set of bistochastic operators on the classical $L^1(X,\mu)$ is denoted $\mathfrak{B}(L^1(X,\mu))$.

Theorem

If $\nu = \mu I_{\mathcal{H}}$ for some finite, positive measure μ, then every $B \in \mathfrak{B}(L^1(X,\mu))$ extends to a bistochastic operator in $\mathfrak{B}(X,\nu)$ by the formula

$$B\,(fA) = B(f)A, \quad \forall f \in L^1(X,\mu),\ A \in \mathcal{B}(\mathcal{H}).$$

We will refer to the extension developed in the above theorem by B as well and the set of such bistochastic operators as $\mathfrak{B}(L^1(X,\mu))$ still. We have no example of a bistochastic operator on $L^1_{\mathcal{H}}(X,\mu I_{\mathcal{H}})$ that does not arise in this way.

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Every bistochastic operator is contractive with respect to the $\| \cdot \|_1$-norm.

The set of bistochastic operators on the classical $L^1(X, \mu)$ is denoted $\mathfrak{B}(L^1(X, \mu))$.

Theorem

If $\nu = \mu I_{\mathcal{H}}$ for some finite, positive measure μ, then every $B \in \mathfrak{B}(L^1(X, \mu))$ extends to a bistochastic operator in $\mathfrak{B}(X, \nu)$ by the formula

$$B\,(fA) = B(f)A, \quad \forall f \in L^1(X, \mu),\ A \in \mathcal{B}(\mathcal{H}).$$

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Variants of Multivariate Majorization

Recall that if $f \in L^1_{\mathcal{H}}(X, \mu I)$ and $s \in \mathcal{T}(\mathcal{H})$ then we define $f_s \in L^1(X, \mu)$ by

$$f_s(x) = \mathrm{Tr}(sf(x)) \in L^1(X, \mu).$$

We now introduce several possible majorization partial orders which relate to multivariate majorization

Definition

Suppose $f, g \in L^1_{\mathcal{H}}(X, \mu I)$ and are self-adjoint where μ is a finite, positive, complex measure. We say that

1. $f \prec g$ if there exists a bistochastic operator $B \in \mathfrak{B}(L^1(X, \mu))$ such that $Bg = f$,
2. $f \prec_T g$ if $f_t \prec g_t$ for all $t \in \mathcal{T}(\mathcal{H})_{sa}$, and
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Relating the Three Partial Orders

Proposition

For $f, g \in L^1_{\mathcal{H}}([0,1], \mu I)$ self-adjoint we have that

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A Result of Komiya

Komiya (1983): For $X, Y \in M_{m,n}(\mathbb{C})$, we have that $X \prec Y$ if and only if $\psi(X) \leq \psi(Y)$ for every real-valued, permutation-invariant, convex function ψ on $M_{m,n}(\mathbb{C})$.

(Note: The convex hull of the permutation matrices is the set of bistochastic matrices.)

We use the notation C_ϕ to denote the right-composition operator: $C_\phi(f) = f \circ \phi$, and $\mathcal{P}_{\mathrm{inv}}$ to denote the set of all invertible measure-preserving maps of X, where the measure is understood by context. If $\phi \in \mathcal{P}_{\mathrm{inv}}$ then C_ϕ is a bistochastic operator.

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Proposition

Suppose X is a product of unit intervals and μ is the corresponding product of Lebesgue measures. If B is a bistochastic operator in $\mathfrak{B}(L^1(X,\mu))$ then there exists a sequence of bistochastic operators $B_i \in \mathrm{conv}(C_\phi : \phi \in \mathcal{P}_{\mathrm{inv}})$ such that B_i is WOT-convergent to B. Moreover, $\mathfrak{B}(L^1(X,\mu))$ is WOT-compact and convex.

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A real-valued convex function $\psi : L^1_{\mathcal{H}}(X, \mu I) \to \mathbb{R}$ is said to be *permutation-invariant* if for every $\sigma \in \mathcal{P}_{\text{inv}}$ we have

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