Amenable dynamical systems through Herz-Schur multipliers

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with A. Bearden

arXiv:2004.01271

CMS Summer Meeting, June 8, 2021

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- $C^*_\lambda(G) = \overline{\mathrm{span}}^{\|\cdot\|}\{\lambda(f) \mid f \in L^1(G)\}$ reduced C^*-algebra

Herz–Schur Multipliers

de Cannière–Haagerup '85: A bd cts $h : G \to \mathbb{C}$ is a (CP) **Herz-Schur multiplier** if the map

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5. (Losert–Ruan) Herz-Schur multipliers $= \operatorname{span} \mathcal{P}(G)$.

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$(\alpha, \lambda \otimes 1)$ covariant rep and **reduced crossed product**

$$G \ltimes A = \overline{(\alpha \times \lambda)(C_c(G, A))}^{\|\cdot\|} \subseteq \mathcal{B}(L^2(G))\overline{\otimes}\mathcal{B}(H),$$

for faithful $A \subseteq \mathcal{B}(H)$.

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Definition (McKee–Todorov–Turowska '16)

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Example: If $\xi \in C_c(G, A)$, the function

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$h \in C_b(G, A)$ is of **positive type** (with respect to (A, G, α)) if for every $n \in \mathbb{N}$, and $s_1, \ldots, s_n \in G$, the matrix $[\alpha_{s_i}(h(s_i^{-1} s_j))] \geq 0$.

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If $h \in C_b(G, Z(A))$ is of **positive type** then $h(s)(a) = h(s)a$ defines a CP Herz-Schur multiplier of (A, G, α).

Amenability

Definition (Zimmer '77; Anantharaman–Delaroche '79)

(M, G, α) *is amenable if there exists a projection of norm one* $P : L^\infty(G)\overline{\otimes}M \to M \cong 1 \otimes M$ *for which*

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- $(L^\infty(G/H), G, \lambda)$ is amenable iff H is amenable.

Reiter's properties

Theorem (Anantharaman–Delaroche '87)

*Let G be **discrete**. TFAE:*

1. $\exists$ (h_i) *positive type functions in* $C_c(G, Z(M))$ *such that*
   1. $h_i(e) \leq 1$;
   2. $\lim_i h_i(t) = 1$ *weak**, $t \in G$.
2. $\exists$ (ξ_i) *in* $C_c(G, Z(M))$ *such that*
   1. $\langle \xi_i, \xi_i \rangle \leq 1$;
   2. $\langle \xi_i, (\lambda_t \otimes \alpha_t)\xi_i \rangle \to 1$ *weak**, $t \in G$.
3. $\exists$ (g_i) *in* $C_c(G, Z(M)^+)$, *such that*
   1. $\sum_{s\in G} g_i(s) \leq 1$;
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Question: Does the Theorem hold for G locally compact?

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Obtained for **exact** groups by Buss–Echterhoff–Willet '20.

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Bearden–C. '20: (2) $\Leftrightarrow$ (1) for any G.

Related notion

Definition (Exel '97, Exel–Ng '02)

(A, G, α) has the 1-positive approximation property (AP) if $\exists$ net (ξ_i) $C_c(G, A)$ for which $\|\langle \xi_i, \xi_i \rangle\| \leq 1$ and

$$\langle \xi_i, (1 \otimes f(s))(\lambda_s \otimes \alpha_s)\xi_i \rangle \to f(s), \quad f \in C_c(G, A)$$

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Exel–Ng '02: A nuclear and G discrete, AP $\Rightarrow$ amenability, with equality if, in addition, A is commutative or finite-dimensional.

Relations between notions

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Partially answer (1) and fully answer (2).

AP through Herz-Schur Multipliers

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Assumption $Z(A^{**}) = Z(A)^{**}$ circumnavigates (A, G, α)-version of Godement's theorem.

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Ozawa–Suzuki '20: amenability = AP for all (A, G, α).

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$(C_0(X), G, \alpha)$ *is* ***measurewise amenable*** *if for every quasi-invariant Radon measure* μ *on* X, $(L^\infty(X, \mu), G, \alpha)$ *is amenable.*

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Corollary (Buss–Echterhoff–Willet '20, Bearden–C. '20)

If X and G are locally compact and second countable, X is topologically amenable $\Leftrightarrow$ X is measurewise amenable.

Thank you!

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1. $T(f) \equiv f$ (meaning equal a.e)
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A. and C. Ionescu–Tuclea '61: Such liftings exist.