Basic homotopy lemmas via abstract classification

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Overview

- Basic homotopy lemmas
- Applications
- New basic homotopy lemma

Basic homotopy lemmas

Theorem (Bratteli–Elliott–Evans–Kishimoto ’98, Kishimoto ’98)

Let B be a real rank zero simple unital A$\mathbb{T}$ algebra and let u be a unitary in $C([0,1],B)_{\mathcal{U}} \cap B'$. Then $u(0)$ is homotopic to $u(1)$ in $B_{\mathcal{U}} \cap B'$.

Basic homotopy lemmas

Theorem (Bratteli–Elliott–Evans–Kishimoto '98, Kishimoto '98)

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Some generalizations:

- Matui '10: to real rank zero simple AH algebras with slow dimension growth.
- Lin '10: to tracial rank zero simple C*-algebras B, and more general commutants $B_{\mathcal{U}} \cap A'$ where A is a full AH subalgebra of $B_{\mathcal{U}}$.

Applications

Approximate cohomology vanishing (Kishimoto)

Let B be a unital C^*-algebra and let $\alpha \in \mathrm{Aut}(B)$ be an automorphism with the Rokhlin property. If $u \in U(B_{\mathcal{U}} \cap B')_0$ then there exists $v \in U(B_{\mathcal{U}} \cap B')_0$ such that $u = v\alpha(v^*)$.

Applications

Other applications:

- Used by Gong, Lin, Niu to upgrade approximate classification to asymptotic classification.
- Calculation of $K_1(B_{\mathcal{U}} \cap A')$.

New basic homotopy lemma

Lemma (Theorem?) (CGSTW '26?)

Let:

- A be a separable unital exact C*-algebra satisfying the UCT,
- B be a unital $\mathcal{Z}$-stable C*-algebra such that every quotient has a trace, and with real rank zero,
- $\phi\colon A \to B_{\mathcal{U}}$ be a full, unital, nuclear embedding,
- $u \in C([0,1], B)_{\mathcal{U}} \cap \phi(A)'$ such that $u(0) = 1_{B_{\mathcal{U}}}$.

Then $u(1) = e^{ih}e^{ik}$ for self-adjoint elements $h, k \in B_{\mathcal{U}} \cap \phi(A)'$ of norm $\leq \pi$.

K_1 calculation

Theorem (CGSTW '26?)

Let:

- A be a separable unital exact C*-algebra satisfying the UCT,
- B be a unital $\mathcal{Z}$-stable C*-algebra such that every quotient has a trace, and with real rank zero,
- $\phi\colon A \to B_{\mathcal{U}}$ be a full, unital, nuclear embedding.

Then

$$K_1(B_{\mathcal{U}} \cap \phi(A)') \cong \mathrm{Hom}_{\Lambda}(\underline{K}(SA), \underline{K}(B_{\mathcal{U}})).$$

Main ingredient

Classification of embeddings (CGSTW '26?)

Let:

- A be a separable unital exact C*-algebra satisfying the UCT,
- B be a unital $\mathcal{Z}$-stable C*-algebra such that every quotient has a trace.

Then full unital nuclear *-homomorphisms $A \to B_{\mathcal{U}}$ are classified by $\underline{K}T_u(\cdot)$.

Apply to $C(\mathbb{T}, A)$ in place of A.