HIGH-DIMENSIONAL $\mathcal{Z}$-STABLE AH ALGEBRAS

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ABSTRACT. It is shown that C^*-algebras of the form $C(X,U)$, where U is a UHF algebra, is not an inductive limit of matrix algebras over commutative C^*-algebras of topological dimension less than that of X. This is in sharp contrast to dimension-reduction phenomenon in (i) simple inductive limits of such algebras, where classification implies low-dimensional approximations, and (ii) when dimension is measured using decomposition rank, as the author and Winter proved that $\mathrm{dr}(C(X,U)) \leq 2$.

1. INTRODUCTION

Consider C^*-algebras that take the form of a direct sum of algebras of continuous functions from a topological space to a matrix algebra; call this class $\mathcal{C}$. Now consider the class $\mathrm{A}\mathcal{C}$ of algebras that are inductive limits of algebras in $\mathcal{C}$. Such algebras, including AF, AI, A$\mathbb{T}$, and (some) AH algebras, have arisen naturally, for instance, as crossed products of the Cantor set or the circle by minimal homeomorphisms. However, the present purpose of considering this class of C^*-algebras is as a test case for phenomena among less-understood finite nuclear C^*-algebras.

A mixture of classification and other arguments has shown that there is a dichotomy amongst the simple C^*-algebras in $\mathrm{A}\mathcal{C}$, dividing them into algebras of low and high topological dimension. Classification arguments, on the one hand, show that for a simple algebra in $\mathrm{A}\mathcal{C}$, if it is an inductive limit of building blocks (in $\mathcal{C}$) with bounded topological dimension (or even "slow dimension growth"), or if it is $\mathcal{Z}$-stable, then it is an inductive limit of algebras in $\mathcal{C}$ with topological dimension at most three [3, 4, 6]. By a general $\mathcal{Z}$-stability theorem of Winter [12], it follows that this is also the case for simple algebras in $\mathrm{A}\mathcal{C}$ with finite nuclear dimension (or decomposition rank). Simple algebras in $\mathrm{A}\mathcal{C}$ without these low-dimensional building blocks were found by Villadsen [11], and further analyzed by Toms and Winter [10].

The dimension reduction alluded to above owes itself to (i) simplicity and (ii) dimension-reducing effects of tensoring with $\mathcal{Z}$ (note that one can show, without classification, that slow dimension growth implies

$\mathcal{Z}$-stability; see [8, 9, 12]). Villadsen's high-dimensional algebras show that simplicity alone does not produce dimension reduction. Moreover, the named author and Winter showed that dimension-reduction, in terms of decomposition rank (and therefore also nuclear dimension), does occur even for nonsimple, $\mathcal{Z}$-stable algebras in A$\mathcal{C}$: if $A \in \mathrm{A}\mathcal{C}$ then $\mathrm{dr}(A \otimes \mathcal{Z}) \leq 2$ [7].

The main result here is that certain non-simple $\mathcal{Z}$-stable algebras in A$\mathcal{C}$ (namely, algebras of the form $C(X, U)$ where U is a UHF algebra) cannot be approximated by low-dimensional algebras in $\mathcal{C}$.[1] This result clarifies the role played by simplicity in the classification results of [3, 4, 6]. It also provides the first example of an approximately homogeneous algebra A with finite decomposition rank, which cannot be approximated by homogeneous algebras with finite decomposition rank.

In [5], Kirchberg and Rørdam devised a way to show that for any commutative C^*-algebra C, $C \otimes 1_{\mathcal{O}_2}$ can be approximated within $C \otimes \mathcal{O}_2$ by commutative C^*-algebras with one-dimensional spectrum. They use this result to show that a number of strongly purely infinite, non-simple algebras are approximated by algebras in $\mathcal{C}$ with one-dimensional spectrum; the result also plays a crucial role in the dimension-reduction result of [7]. Their result rests mainly on the fact that $\mathcal{O}_2$ has trivial K-theory, which is closely tied to the existence of an $\mathcal{O}_2$-relativized retract $D^2 \to S^1$. It is not difficult to adapt their argument (as we do in Section 5) to show that if A is such that we can solve

$$(1.1) \qquad \begin{array}{ccc} C(S^{n-1}) & \xrightarrow{\ \exists\ } & C(D^n, A) \\ & \searrow{\scriptstyle \mathrm{id} \otimes 1_A} & \big\downarrow{\scriptstyle f \mapsto f|_{S^{n-1}}} \\ & & C(S^{n-1}, A), \end{array}$$

then for any n-dimensional space X, $C(X) \otimes 1_A$ can be approximated in $C(X, A)$ by commutative C^*-algebras with $(n-1)$-dimensional spectrum. Our main result arises by showing that the converse is true: if $C(X) \otimes 1_A$ can be approximated in $C(X, A)$ by $(n-1)$-dimensional commutative algebras then there is an A-relativized retract of D^n onto S^{n-1}.

The proof is broken into steps (and indeed we state the main theorem as a number of equivalent conditions on A); in Section 3, we show that these $(n-1)$-dimensional approximants imply an approximate A-relativized retract of D^n onto S^{n-1} (i.e. that we can solve (1.1) point-norm approximately), while in Section 4, we show that the approximate and exact versions are equivalent.

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[1]In work in progress, we are proving a stronger version of this result, where we replace $\mathcal{C}$ by the class of subhomogeneous C^*-algebras.

Let us introduce our notation precisely before clearly stating the main result. For a C^*-algebra A, a C^*-subalgebra B, and a class $\mathcal{S}$ of C^*-algebras, we say that B is **locally approximated** in A by C^*-algebras in $\mathcal{S}$ if the following holds: for every finite subset $\mathcal{F}$ of B and every $\epsilon > 0$, there exists a C^*-subalgebra C of A such that $C \in \mathcal{S}$ and $\mathcal{F} \subset_\epsilon C$. Note that if A is an inductive limit of algebras in $\mathcal{S}$ (or an inductive limit of inductive limits of algebras in $\mathcal{S}$, etc.) then it is locally approximated by algebras in $\mathcal{S}$.

If B is a commutative C^*-algebra, the value of its decomposition rank and nuclear dimension coincide (and are equal to the covering dimension of X if $B \cong C_0(X)$, at least when B is separable); we use the term **topological dimension** to refer to this value.

Theorem 1.1. *Let A be a unital C^*-algebra and let $n \in \mathbb{N}$. TFAE:*

(i) *For every n-dimensional compact Hausdorff space X, $C(X)$ is approximated in $C(X, A)$ by commutative C^*-algebras of topological dimension at most $n - 1$;*

(ii) *$C(D^n)$ is approximated in $C(D^n, A)$ by commutative C^*-algebras of topological dimension at most $n - 1$;*

(iii) *For every n-dimensional finite CW complex X, the inclusion $C(X) \to C(X, A)$ approximately factors through C^*-algebras of the form $C(Y)$, where Y is a finite CW complex of dimension at most $n - 1$;*

(iv) *For any finite set $\mathcal{F} \subset C(S^{n-1})$ and $\epsilon > 0$, there exists a $*$-homomorphism $\phi : C(S^{n-1}) \to C(D^n, A)$ such that*

$$a \approx_\epsilon \phi(a)|_{S^{n-1}}$$

for all $a \in \mathcal{F}$;

(v) *The inclusion map $C_0(\mathbb{R}^{n-1}) \to C_0(\mathbb{R}^{n-1}, A)$ is nullhomotopic (within the space of $*$-homomorphisms $C_0(\mathbb{R}^{n-1}) \to C_0(\mathbb{R}^{n-1}, A)$).*

If $A \cong A \otimes A$ (using any tensor norm), then these are also equivalent to the statements (i) and (iii) with the words "n-dimensional" removed; that is, that for any Hausdorff space X, $C(X)$ is approximated in $C(X, A)$ by commutative C^*-algebras of topological dimension at most $n - 1$, and if X is a finite CW complex, then the inclusion $C(X) \to C(X, A)$ approximately factors through commutative C^*-algebras of topological dimension at most $n - 1$.

By an easy K-theoretic obstruction to (v), we obtain:

Corollary 1.2. *If U is a UHF algebra then $C(D^n, U)$ is not locally approximated by algebras in $\mathcal{C}$ of topological dimension less than n.*

In the remainder of this note, we prove the various implications of Theorem 1.1, as outlined above. (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are all obvious.

2. (II) AND COVERING DIMENSION

We show here that for each space X, another, somewhat technical, covering-dimension-related condition is equivalent to the condition in Theorem 1.1 (ii).

Let X be a compact metric space and let A be a C^*-algebra. Let $\mathcal{U}$ be an open cover of X. Following T-Winter, an $(n+1)$-colourable partition of unity subordinate to $\mathcal{U}$ means positive elements $b_j^{(i)} \in C(X, A)$ for $i = 0, \ldots, n, j = 1, \ldots, r$, such that:

(i) for each i, the elements $b_1^{(i)}, \ldots, b_r^{(i)}$ are pairwise orthogonal,
(ii) for each i, j, the support of $b_j^{(i)}$ is contained in some open set in the given cover $\mathcal{U}$, and
(iii) $\sum_{i,j} b_j^{(i)} = 1$.

Proposition 2.1. *Let X be a compact metric space, let $n \in \mathbb{N}$, and let A be a unital C^*-algebra. TFAE:*

(a) $C(X)$ is approximated in $C(X, A)$ by abelian C^*-algebras whose spectrum has dimension at most n;
*(b) For every open cover $\mathcal{U}$ of X, there exists a **commuting** $(n+1)$-colourable partition of unity subordinate to $\mathcal{U}$.*

Remark 2.2. In T-Winter Proposition 3.2, it is shown that the existence of $(n+1)$-colourable approximate partitions of unity in $C(X, A)$ is equivalent to $\dim_{\mathrm{nuc}}(C(X) \subset C(X, A)) \leq n$. (b) is an notable strengthening, in that the partition of unity is asked to be commuting. (Note that whether or not the partition of unity is approximate is moot, since any commuting approximate partition of unity can be turned into a commuting exact partition of unity by functional calculus.) Theorem 1.1 shows that in many cases, (b) is not equivalent to the weaker condition of non-commuting approximate partitions of unity.

Proof. (a) $\Rightarrow$ (b): Let $\mathcal{F}$ be a finite partition of unity such that, for each $f \in \mathcal{F}$, there exists $U_f \in \mathcal{U}$ such that $\mathrm{supp}\, f \subset U_f$. Use (b) to obtain a space Y of dimension at most n and a $*$-homomorphism $\phi : C(Y) \to C(X, A)$ such that $\mathcal{F} \subset_\epsilon \phi(C(Y))$ (for some sufficiently small ϵ). For each $f \in \mathcal{F}$, let $g_f \in C(Y)$ be such that $f \approx \phi(g_f)$. By functional calculus, we may assume that $\mathrm{supp}\, \phi(g_f) \subset U_f$. Note that $1 \approx_{|\mathcal{F}|\epsilon} \sum_{f \in \mathcal{F}} \phi(g_f)$.

Set $Y' := \{y \in Y : \sum_{f \in \mathcal{F}} g_f(y) \geq 1 - 2|\mathcal{F}|\epsilon\}$. It follows that $\ker \phi = C_0(Z)$ where $Z \cap Y' = \varnothing$.

Since Y has dimension at most n, let $(a_j^{(i)})$ be $(n+1)$-colourable, subordinate to $\{g_f^{-1}((0, \infty)) : f \in \mathcal{F}\}$, such that

$$\sum_{i,j} a_j^{(i)}|_{Y'} = 1'_Y.$$

Set

$$b_j^{(i)} := \phi(a_j^{(i)}).$$

This is $(n+1)$-colourable since $(a_j^{(i)})$ is. It is subordinate to $\mathcal{U}$, since if the support of $a_j^{(i)}$ is contained in $g_f^{-1}((0,\infty))$ then the support of $b_j^{(i)}$ is contained in U_f. Finally, $1 - \sum_{i,j} a_j^{(i)} \in C_0(Y \backslash Y') \subseteq C_0(Z)$, $\sum_{i,j} b_j^{(i)} = 1$.

(b) $\Rightarrow$ (a): This is clear, since the universal C^*-algebra generated by a commuting, $(n+1)$-colourable partition of unity $(b_j^{(i)})_{i=0,\dots,n-1,j=1,\dots,r}$ is $C(Z)$ where Z is a certain n-dimensional simplicial complex. $\qquad\square$

3. (II) $\Rightarrow$ (IV)

In the following, the n-nerve of a finite open cover $\mathcal{U}$, means the following (abstract) simplicial complex on the vertex set $\mathcal{U}$,

$$N_n(\mathcal{U}) := \{ \{U_1, \dots, U_k\} \subseteq \mathcal{U} \mid k \le n \text{ and } U_1 \cap \cdots \cap U_k \ne \varnothing \}.$$

(That is, it is the n-skeleton of the nerve of $\mathcal{U}$.) We denote its geometric realization $|N_n(\mathcal{U})|$. For $U \in \mathcal{U}$, we let $\mathrm{Star}(U)$ denote the star around U, as an open subset of $|N_n(\mathcal{U})|$. (It is the union of all the interior parts of the realization of faces containing U.)

Lemma 3.1. *Let $\mathcal{U}$ be an open cover of S^{n-1}. Then there exists an open cover $\mathcal{V}$ of D^{n+1} and a continuous map $\alpha : |N_{n+1}(\mathcal{V})| \to S^{n-1}$ such that the following holds: If $U \in \mathcal{U}$ and $V \in \mathcal{V}$ are such that $U \cap V \ne \varnothing$, then*

$$\alpha(\mathrm{Star}(V)) \subseteq U.$$

Proof. First, let $\mathcal{W}$ be an $(n+1)$-colourable refinement of $\mathcal{U}$, such that, for any $W_1, \dots, W_m \in \mathcal{W}$, if

$$W_1 \cap \cdots \cap W_m \ne \varnothing$$

then $W_1 \cup \cdots \cup W_m$ is contained in some set $U \in \mathcal{U}$. (This can be done by taking a barycentric refinement, see eg. [1, VIII.3], and then an $(n+1)$-colourable refinement of that.)

Let $(f_W)_{W \in \mathcal{W}}$ be a partition of unity such that the support of f_W is contained in W for all $W \in \mathcal{W}$. This induces a map $\hat{\alpha} : N_{n+1}(\mathcal{W}) \to S^n$ such that, if $x \in S^n$, $U \in \mathcal{U}$, $W \in \mathcal{W}$ are such that $x \in U \cap W$, then

(3.1) $$\hat{\alpha}(\mathrm{Star}(W)) \subseteq U.$$

Now, view D^{n+1} as CS^n, and let $\pi : S^n \times [0,1] \to D^{n+1}$ be the quotient map. For $W \in \mathcal{W}$, set

$$v_0(W) := \pi(V \times [0, 1/2)), \quad v_1(W) := \pi(V \times (0,1)).$$

Then define

$$\mathcal{V} := \{v_i(W) \mid i = 0, 1, W \in \mathcal{W}\} \cup \{\pi(S^n \times (1/2, 1])\}$$

Note that $v_0(W) \cap \pi(S^n \times (1/2, 1]) = \varnothing$, so that if $\{V_1, \ldots, V_k\} \in N_{n+1}(\mathcal{V})$ then either:

(a) Each V_j is of the form $v_i(W)$ for some $i = 0, 1$ and $W \in \mathcal{W}$; or
(b) Some V_j is equal to $\pi(S^n \times (1/2, 1])$ and for every other j, V_j is of the form $v_1(W)$ for some $W \in \mathcal{W}$.

Let N' be the subcomplex consisting of simplices of the first type; then the map $v_i(W) \to W$ induces a map $N' \to N_{n+1}(\mathcal{W})$, and thereby a map $\beta : |N'| \to |N_{n+1}(\mathcal{W})|$. We set

$$\alpha|_{|N'|} := \hat\alpha \circ \beta : |N'| \to S^n.$$

Then, since $|N_{n+1}(\mathcal{V})|$ is n-dimensional, we may extend this map to all of $|N_{n+1}(\mathcal{V})|$.

Now, if $x \in S^n$, $U \in \mathcal{U}$, $V \in \mathcal{V}$ are such that $x \in U \cap V$, then by definition of $\mathcal{V}$, $V = v_0(W)$ for some W with $x \in W$. Consequently, we see that $\text{Star}(V) \subseteq |N'|$, and $\beta(\text{Star}(V)) = \text{Star}(W)$ in $|N_{n+1}(\mathcal{W})|$. By (3.1) and the definition of α, it follows that $\alpha(\text{Star}(W)) \subseteq U$, as required. $\qquad\square$

The following proposition, together with Proposition 2.1, establishes (ii) $\Rightarrow$ (iv) of Theorem 1.1.

Proposition 3.2. *Let A be a unital C^*-algebra and let $n \in \mathbb{N}$. If $C(D^n)$ is approximated in $C(D^n, A)$ by commutative C^*-algebras whose spectrum has dimension at most $(n-1)$, then (iv) of Theorem 1.1 holds.*

Proof. Let $\mathcal{F} \subset C(S^{n-1})$ and $\epsilon > 0$ be given, as in (iv) of Theorem 1.1. Let $\mathcal{U}$ be an open cover of S^{n-1} such that for every $a \in \mathcal{F}$, $U \in \mathcal{U}$ and $x, y \in U$, $\|a(x) - a(y)\| < \epsilon$. Let $\mathcal{V}$ be an open cover of D^n and $\alpha : |N_n(\mathcal{V})| \to S^{n-1}$ provided by Lemma 3.1. By (ii) and Proposition 2.1, let $(b_j^{(i)})_{i=0,\ldots,n;\ j=1,\ldots,r}$ be an n-colourable commuting partition of unity subordinate to $\mathcal{V}$. For each i, j, let $V_j^{(i)} \in \mathcal{V}$ be a set containing the support of $b_j^{(i)}$.

Consider the nerve N of $(b_j^{(i)})$ (by which I mean, the nerve of the cozero sets of the $b_j^{(i)}$'s, as subsets of some topological space Z such that $C^*(\{b_j^{(i)}\}) = C(Z)$); we shall identify the vertices of this simplicial complex with the elements $b_j^{(i)}$. This nerve comes with a canonical map $\psi : C(N) \to C^*(\{b_j^{(i)}\})$. If $\{b_{j_1}^{(i_1)}, \ldots, b_{j_k}^{(i_k)}\}$ is a simplex in N then evidently $k \leq n + 1$ (by $(n + 1)$-colourability), and

$$V_{i_1, j_1} \cap \cdots \cap V_{i_k, j_k} \neq \varnothing.$$

Thus, $b_j^{(i)} \mapsto V_j^{(i)}$ induces a simplicial map from N to $N_{n+1}(\mathcal{V})$, and thereby a continuous map $\beta : |N| \to |N_{n+1}(\mathcal{V})|$.

Define $\phi : C(S^{n-1}) \to C(D^n, A)$ to be the following composition

$$C(S^{n-1}) \xrightarrow{f \mapsto f \circ \alpha} C(|N_n(\mathcal{V})|) \xrightarrow{f \mapsto f \circ \beta} C(|N|) \xrightarrow{\psi} C^*(\{b_j^{(i)}\}) \subseteq C(D^n, A).$$

Let us now show that $\|a - \phi(a)|_{S^{n-1}}\| \leq \epsilon$ for $a \in \mathcal{F}$.

Certainly, let $x \in S^{n-1}$ and $a \in \mathcal{F}$. Let $U \in \mathcal{U}$ contain x. Let the kernel of $\mathrm{ev}_x \circ \phi$ be $C_0(S^{n-1} \setminus Y)$, where $Y \subseteq S^{n-1}$ is closed; thus, $\mathrm{ev}_x \circ \phi$ can be viewed as a representation of $C(Y)$. Using the fact that $a(x) \in \mathbb{C}$ and ϕ is unital, we see that $\|a(x) - \phi(a)(x)\| \leq \sup_{y \in Y} \|a(x) - a(y)\|$. We shall show that $Y \subseteq U$, so that we may conclude that $\|a(x) - \phi(a)(x)\| \leq \epsilon$.

If the kernel of $\mathrm{ev}_x \circ \psi$ is $C_0(|N| \setminus Z)$ then, by the definition of ψ, we can see that Z is contained in the union of stars about vertices $b_j^{(i)}$ for which $b_j^{(i)}(x) \neq 0$, so that $V_j^{(i)} \cap U \neq \varnothing$. Hence,

$$\beta(Z) \subseteq \bigcup \{\mathrm{Star}(V) \mid V \in \mathcal{V}, V \cap U \neq \varnothing\}.$$

Consequently,

$$Y = \alpha \circ \beta(Z) \subseteq \bigcup \{\alpha(\mathrm{Star}(V)) \mid V \in \mathcal{V}, V \cap U \neq \varnothing\},$$

and by Lemma 3.1, this is contained in U, as required. $\qquad \square$

4. (IV) $\Leftrightarrow$ (V)

The following is a fact from the theory of ANR's.

Lemma 4.1. [?, Theorem 1.1] *Let $n \in \mathbb{N}$. There is a finite set $\mathcal{F} \subset C(S^n)$ and $\epsilon > 0$ such that the following holds: If A is a commutative C^*-algebra and $\phi_0, \phi_1 : C(S^n) \to A$ are $*$-homomorphisms which satisfy*

$$\phi_0(a) \approx_\epsilon \phi_1(a)$$

for all $a \in \mathcal{F}$, then ϕ_0 and ϕ_1 are homotopic, i.e. there is a $*$-homomorphism $\phi : C(S^n) \to C([0,1], A)$ such that

$$\phi_i = \mathrm{ev}_i \circ \phi$$

for $i = 0, 1$.

The following establishes (iv) $\Leftrightarrow$ (v).

Proposition 4.2. *Let A be a unital C^*-algebra and let $n \in \mathbb{N}$. TFAE:*

(a) *For any finite set $\mathcal{F} \subset C(S^{n-1})$ and $\epsilon > 0$, there exists a $*$-homomorphism $\phi : C(S^{n-1}) \to C(D^n, A)$ such that*

$$a \approx_\epsilon \phi(a)|_{S^{n-1}}$$

for all $a \in \mathcal{F}$;

(b) *There exists a $*$-homomorphism $\phi : C(S^{n-1}) \to C(D^n, A)$ such that*

$$a = \phi(a)|_{S^{n-1}}$$

for all $a \in C(S^{n-1})$;

(c) *The inclusion map $C(S^{n-1}) \to C(S^{n-1}, A)$ is homotopic, within the space of $*$-homomorphisms $C(S^{n-1}) \to C(S^{n-1}, A)$, to a point-evaluation.*

(d) *The inclusion map $C(S^{n-1}) \to C(S^{n-1}, A)$ is nullhomotopic, within the space of $*$-homomorphisms $C(S^{n-1}) \to C(S^{n-1}, A)$.*

Proof. (a) $\Rightarrow$ (b) Let $\mathcal{F} \subset C(S^{n-1})$ and $\epsilon > 0$ be given by Lemma 4.1. Use (i) to get a $*$-homomorphism

$$\phi_0 : C(S^{n-1}) \to C(D^n, A)$$

such that $\phi_0(a)|_{S^{n-1}} \approx_\epsilon a$ for all $a \in \mathcal{F}$. Note that

$$B := C^*(C(S^{n-1}) \cup \phi_0(C(S^{n-1}))|_{S^{n-1}})$$

is a commutative subalgebra of $C(S^{n-1}, A)$. Therefore by Lemma 4.1, there exists $\psi : C(S^{n-1}) \to C([0,1], B)$ such that

(4.1) $\mathrm{ev}_0 \circ \psi = \phi_0|_{S^{n-1}}$

and $\mathrm{ev}_1 \circ \psi$ is equal to the trivial inclusion $C(S^{n-1}) \subseteq B$.

D^n is homeomorphic to $(D^n \cup [0,1] \times S^{n-1})/\sim$, where $\sim$ identifies each point of $\partial D^n \cong S^{n-1}$ with the corresponding point of $\{0\} \times S^{n-1}$. Viewing D^n in this way, we see that (4.1) ensures that $\phi \oplus \psi : C(S^{n-1}) \to C(D^n, A) \oplus C([0,1] \times S^{n-1}, A)$ induces a map $\phi : C(S^{n-1}) \to C(D^n, A)$ such that $\phi(a)|_{S^{n-1}} = \psi(a)|_{\{1\} \times S^{n-1}} = a$, as required.

(b) $\Rightarrow$ (a) is obvious.

(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) can be seen using well-known topological arguments. $\square$

5. (IV) $\Rightarrow$ (III)

Proof of (iv) $\Rightarrow$ (iii). This is essentially contained in the proof of [5, Proposition 3.5].

Assume that (iv) holds.

Given a finite subset $\mathcal{F}$ of $C(X)$ and $\epsilon > 0$, let us take a CW decomposition of X so that each $a \in \mathcal{F}$ varies by at most ϵ on each cell of the complex. We may view this decomposition as a canonical surjection

$$\alpha : \coprod_{k=1}^{r} D_k \to X,$$

where each D_k is homeomorphic to a disc of dimension at most n, and the restriction of α to $\bigcup D_k^\circ$ is one-to-one. Composition with α provides an injective $*$-homomorphism $C(X) \to C(\coprod_{k=1}^{r} D_k)$; we will identify $C(X)$ with its image under this map. For each k, define $Y_k \subseteq D_k$ and $\phi_k : C(Y_k) \to C(D_k, A)$ as follows: If D_k has dimension n, set $Y_k := \partial D_k$ and let ϕ_k be as given by (ii) of Proposition 4.2. Otherwise, set $Y_k := D_k$ and let $\phi_k : C(D_k) \to C(D_k, A)$ be the inclusion.

Note that in both cases, we have $\partial D_k \subseteq \partial Y_k$ and $\phi_k(a)|_{\partial D_k} = a|_{\partial D_k}$ for all $a \in C(D_k)$. Set $\hat{\phi} = \bigoplus \phi_k : C(\coprod_{k=1}^{r} Y_k) \to C(\bigcup_{k=1}^{r} D_k)$, and set

$Y = \alpha(\coprod Y_k)$. Since $\hat{\phi}(a)|_{\bigcup \partial D_k} = a|_{\bigcup \partial D_k}$, we see that for $a \in C(Y) \subseteq C(\coprod Y_k)$,

$$\hat{\phi}(a) \in C(X, A) \subseteq C(\coprod D_k, A).$$

Hence, $\phi := \hat{\phi}|_{C(Y)}$ is a map from $C(Y)$ to $C(X, A)$. For $x \in X$, the kernel of $\mathrm{ev}_x \circ \phi$ is $C_0(Y \backslash K)$ where

$$K \subseteq \bigcup \{\alpha(D_k) \mid x \in \alpha(D_k)\}.$$

Since each $a \in \mathcal{F}$ varies by at most ϵ on each $\alpha(D_k)$, we see that

$$\phi(a|_Y)(x) \approx_\epsilon a(x).$$

$\square$

Proof of the last sentence of Theorem 1.1. The above argument, and induction, shows that if Theorem 1.1 (iv) holds, then for any $m \geq n$, the inclusion $C(D^m) \to C(D^m, A^{\otimes m})$ can be approximately factorized as

$$C(D^m) \xrightarrow{\psi} C(\Gamma) \xrightarrow{\phi} C(D^m, A^{\otimes(m-n+1)}),$$

where Γ has dimension at most $n - 1$ and $\psi\phi(f)|_{S^{m-1}} = f|_{S^{m-1}}$. We may identify A^{m-n+1} with A, so that for any $m \in \mathbb{N}$, the inclusion $C(D^m) \to C(D^m, A^{\otimes m})$ can be approximately factorized as

$$C(D^m) \xrightarrow{\psi} C(\Gamma) \xrightarrow{\phi} C(D^m, A^{\otimes(m-n+1)}),$$

where Γ has dimension at most $n - 1$ and $\psi\phi(f)|_{S^{m-1}} = f|_{S^{m-1}}$ (this factorization is trivial if $m \leq n - 1$). Then, a patching-together argument as in the above proof shows that the (iii) holds with the "n-dimensional" removed, and again, this obviously implies that (i) holds with the words "n-dimensional" removed. $\square$

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